THE ROORKEE MANUAL OF APPLIED MECHANICS;

STABILITY OF STRUCTURES

AND THE

GRAPHIC DETERMINATION OF LINES OF RESISTANCE.

VOL. II.

BY

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PREFACE TO VOL. II.

As the circumstances under which the Second Edition of this Manual of Applied Mechanics was undertaken are fully set forth in the Preface to Vol. I., it only remains for me to make a few brief comments on the contents of this Volume, and explain the delay which has arisen in its production.

With regard to the former, since the advantages of the graphic treatment of Engineering problems over the analytic are now so universally recognised, it has been my chief aim to provide the Student with a graphic method which, while being easy of application, would enable him to deal intelligently with problems not of ordinary occurrence, to treat which, moreover, by analytical methods would involve a practical acquaintance with the Infinitesimal Calculus, such as but few young men just entering the Civil Engineering Profession possess. This aim will, I think, be found to be met by a knowledge of the Equilibrium Polygon in its varied applications. It is not my wish to substitute the graphic method herein described, and which, I believe, is generally attributed to Professor Culmann of Zürich, for the analytic one explained in Vol. I., but to offer it to the Student as an additional and powerful instrument for the solution of a large class of Engineering problems.

For the illustrations of the graphic method to be found in the following pages, I am mainly indebted to *Chalmers' Graphical Determination of Forces in Engineering Structures* and *Dubois' Graphical Statics*, and for the descriptions of the various kinds of bridges to *Claxton Fidler's Practical Treatise on Bridge Construction.* I am also largely indebted to *Cols. Wray and Seddons' Instruction in Construction*, portions of which were formerly included in the Students' prescribed course of study, and permission for

the free use of which was kindly obtained from the Controller of Her Majesty's Stationery Office; to *Ritter's Iron Bridges and Roofs*; and to *Sir B. Baker's Long Span Railway Bridges.* Other sources of information will, I think be found to be duly acknowledged in the text.

The delay which has occurred in the publication of this Volume is due to the fact that its preparation had to be carried on as opportunity might offer in the midst of my other more pressing and regular duties during the period of my service in England, and also to the fact that the manuscript had to be sent to Roorkee to be printed and published, &c.

In the Preface to Vol. I., it was my pleasure to acknowledge the valuable assistance rendered to me by a brother officer whom death has recently removed in the prime of life. In again acknowledging that help I desire to record my high appreciation of the natural abilities possessed by Capt. E. D. Bullen, R.E., and of the keen interest which he took in all that concerned the education and welfare of the Students committed to his charge.

My thanks are largely due to Mr. Robey, Superintendent of the College Press, for the care which he has bestowed on the preparation of proof sheets and plates.

J. H. C. HARRISON, LIEUT.-COL., R.E.

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ROORKEE TREATISE.

APPLIED MECHANICS.

ON THE STABILITY OF STRUCTURES AND THE GRAPHIC DETERMINATION OF LINES OF RESISTANCE.

PART I.

CHAPTER I.

THE EQUILIBRIUM OF STRUCTURES.

1. In Part I., Chap. V., and Part II., Chap. XII., of Vol. I. of this Manual, a method is fully described whereby the stresses developed in the different parts of roof trusses and girders subjected to normal and vertical loads may be graphically determined; in the following pages the same principle will be applied to the measurement of the Stability of Structures generally.

2. The following definitions are quoted from Professor Rankine's Manual of Applied Mechanics (Third Edition), and the paging noted in this Chapter, unless otherwise stated, has reference to that Manual :—

"A *structure* consists of two or more solid bodies called its *pieces*, which touch each other and are connected at portions of their surfaces called *joints*.

"The term '*stability*' as applied to the condition of a body forming part of a structure 'is the property it possesses' of *remaining in equilibrio* without sensible deviation of position, notwithstanding certain deviations of the load, or externally applied force, from its mean amount or position (pages 128 and 129);" in other words, it is its property of

resisting displacement of its pieces. The stability of structures of all kinds, whether walls, suspension bridges, arches, braced girders, &c., is determinable on similar principles.

3. Now *the conditions of equilibrium of a structure* are three in number, viz., (page 129):—

I.—*That the system of forces exerted on the whole structure by external bodies shall balance each other*, including, of course, the resistances at the point or points of support of the structure.

II.—*That the forces exerted on each piece of the structure shall balance each other.*

III.—*That the forces on each of the parts into which the pieces of the structure can be conceived to be divided shall balance each other.*

Of these three conditions, the *first two* only have reference *to the stability of a structure*, the *third* refers to its *strength and stiffness.*

4. A structure which is deficient in stability gives way by the displacement of its pieces from their proper positions; one deficient in strength by the rupture or disfigurement of its pieces. The fulfilment of all three conditions is, of course, necessary to the permanence of a structure "not only under one amount and distribution of the load, but under all variations of the load both as to amount and distribution which can possibly occur in the use of the structure" (pages 129 and 130).

5. Now "the mode of distribution of the intensity of the load upon a given piece of a structure affects its strength and stiffness only; so far as its *stability* alone is concerned, it is sufficient to know the magnitude and position of the *resultant* of that load which * * * may then be treated as a single force.

"In like manner, when stability only is in question, it is sufficient to consider the position and magnitude of the *resultant* of the resistance or stress exerted between two pieces of a structure at the joint where they meet, and to treat that resultant as a single force. The point where its line of action traverses the joint is called the *centre of resistance* of that joint; and a *line of resistance* is a line, straight, angular, or curved, traversing the centres of resistance of all the joints of a structure" (page 131).

6. "*Joints* may be divided into three classes—

1. "*Frame-work joints* are such as occur in carpentry, in frames of metal bars, and in structures of ropes and chains, fixing the ends of two or more pieces together, but offering little or no resistance to change in the relative angular position of those pieces. *In a joint of this class the*

centre of resistance is at the middle of the joint, and does not admit of any variation of position consistently with security.

2. "*Blockwork joints* are such as occur in masonry and brickwork, being plane or curved surfaces of contact, of considerable extent as compared with the dimensions of the pieces which they connect, capable of resisting a thrust more or less oblique, * * * but not of resisting a pull of sufficient intensity to be taken into account in practice. *In such joints the position of the centre of resistance may be varied within certain limits.*

3. "*Fastened joints*, at which, by means of some strong cement, or of bolts, rivets, or other fastenings, two pieces are so connected that the joint fixes their relative angular position and is capable of resisting a pull as well as a thrust" (pages 131 and 132).

7. The term "*Frame* is used to denote a structure composed of bars, rods, links, or cords, attached together or supported by *joints of the first class*, the centre of resistance being at the middle of each joint, and the line of resistance consequently a polygon whose angles are at the centres of the joints" (page 132).

8. A continuously loaded structure, then, as far as its stability only is concerned, may always be reduced to the case of one subjected to the action of detached loads, by substituting for the continuous loads borne between the several joints the respective resultants of those loads acting at their respective centres of action; and, moreover, by supposing these several detached loads to be entirely and proportionately distributed at the joints, this case may, if necessary, be still further reduced to that of a polygonal frame loaded at the joints only. With this premise the following Theorem may be stated as generally applicable:—

9. "If lines radiating from a point be drawn parallel to the lines of resistance of the bars of a polygonal frame, then the sides of any polygon whose angles lie in those radiating lines will represent a system of forces, which, being applied to the joints of the frame, will balance each other, each such force being applied to the joint between the bars whose lines of resistance are parallel to the pair of radiating lines that enclose the side of the polygon of forces, representing the force in question. Also, the lengths of the radiating lines will represent the stresses along the bars to whose lines of resistance they are respectively parallel" (page 140).

10. "An open polygon, consisting of ties, is called by mathematici-

ans a *funicular polygon*, because it may be made of ropes" (page 141). The term "equivalent funicular polygon" might for convenience be applied to a similar imaginary structure consisting of struts, but we shall employ the term *equilibrium polygon* (or resistance polygon) *as being applicable to one supposed to consist either of ties or struts.*

For instance, suppose ABCDEFGH (*Figs.* 2 and 3, *Plate* I.) to represent a polygonal frame. Take any point P, as pole, and draw a series of straight lines 1, 2, 3, 4, 5, 6, 7, 8 (*Fig.* 1) parallel respectively to the sides of the frame so numbered in *Figs.* 2 and 3. Then the sides of *any* polygon *abcdefgha* whose angles lie *in* the radiating lines will represent a system of forces which, being applied to the joints of the frame, will be balanced by the forces represented by the straight lines 1, 2, 3, 4, 5, 6, 7, 8, each such force being applied to the joint between the bars whose lines of resistance are parallel to the pair of radiating lines that enclose the side of the polygon representing the force in question.

Thus, a force represented on any scale of loads by A′, applied at A, will be balanced by the tension 1 in piece AB, and the compression 8 in AH measured on the same scale; similarly the force C′, applied at C, will balance the two tensions 2 and 3, all being represented on the same scale as that on which the force at A is represented by A′, and so on, the directions of the several forces acting at any one point being such that, if the sides of the corresponding triangle (or polygon) of *Fig.* 1 be carefully followed round with a pencil *in the same order and direction as the forces act*, the pencil will describe a closed figure.

11. A diagram such as that represented in *Fig.* 1, *Plate* I., is called a "force" or "stress" diagram.

12. Similarly, if the frame, *Fig.* 3, be inverted, as shown in *Fig.* 2, and the *same* forces A′, B′, &c., applied at the corresponding joints, the force diagram, *Fig.* 1, will equally apply for determining the *magnitudes* of the stresses in the several bars as in the previous case, but the *nature* of these stresses will become reversed, the tensions in the pieces 1, 2, 3 to 7 changing into compressions, and the compression of piece 8 becoming a tension, as will be evident if the sides of the corresponding "stress" diagram be traced round with a pencil in the manner already indicated.

13. If the applied forces be all parallel, a stress diagram, such as that shown in *Plate* II., *Fig.* 4, will be obtained. *Figs.* 5, 6, and 7 show the corresponding frame diagrams.

14. Thus, considering B′, C′, D′,......G′ to represent externally applied loads and A′, H′ the corresponding resistances at the points of support A and H of the structure, we see that this system of applied loads and corresponding resistances balances, since the magnitudes and directions of its several forces may be represented by the sides of a closed polygon taken in order, and they may be all conceived to act together simultaneously at a single point, viz., the centre of action of the system.

15. It is evident that the straight line *ag* of *Figs.* 1 or 4 represents the *total applied resultant load,* both in direction and magnitude, while *ga* represents the *total resultant resistance* exerted at the points of support of the structure A and H, the portion *gk* being exerted at H and *ka* at A. Further, if the bars AB and HG of the frame (*Figs.* 2 and 3 or 5 and 6) be produced to meet in *p*, and through *p* a straight line *p*K be drawn parallel to *ag*, then must the centre of action of the system of loads lie somewhere on the straight line K*p*, for the three forces *ag*, 1 and 7 meeting in a point balance, as is evident from the triangle P*ag* (*Figs.* 1 and 4).

16. It will be noticed that the straight line P*l*, *Fig.* 1, drawn parallel to the *closing piece* 8 of the equilibrium polygons, *Figs.* 2 and 3, meets the resistances *gh*, *ha*, *Fig.* 1, at their point of intersection *h*; cuts the total resultant load *ga* in the point *k*; and meets the polygon of external loads in *l*. Of these segments, P*h* represents the value of stress 8 in the closing piece; *hk* that of the component of either re-action *gh*, *ha*, resolved parallel to stress 8; *kl*, that of the component of the external loads, resolved likewise in direction of stress 8.

17. Moreover, from statical considerations (taking moments about K, *Figs.* 2 and 3), the straight line AH should be divided at K in the inverse ratio of the resistances acting at the extremities A and H, that is, in the inverse ratio of the lengths of the straight lines *ak*, *kg*, so that $\frac{\mathrm{AK}}{\mathrm{KH}} = \frac{gk}{ka}$ (*Figs.* 1, 2 and 3).

18. Now *Fig.* 3 may be taken to represent the line of resultant active stress of a structure, the pieces of which immediately supporting the load or loads are entirely subjected to tensile strain; that is—

I.—The general case of the Suspension Bridge, including examples of cables and ropes suspended across intervals for boat-bridging or other

purposes, and also of beams of uniform strength whose flanges are of uniform width and thickness and whose webs are curved on the under side.

Fig. 2 may be taken to represent the line of resultant active stress of a structure whose supporting pieces are thrown into compressive strain only; that is—

II.—The general case of the arch, including braced structures, such as the triangular truss, bowstring girder, &c., and beams of uniform strength and uniform flange width and thickness, whose webs are curved on the upper side.

Under this heading come Structures of Uncemented Blockwork generally, to which class the masonry arch belongs, the material of which is incapable of resisting a tensile strain, such as buttresses, retaining walls, factory chimneys, &c. Any portion of the frame shown in *Fig.* 2, as FGH, might, for instance, be taken to represent the resultant line of active thrust of the loads 5, F′, G′, and 8 on the blocks shown by dotted lines in *Fig.* 2*a*, producing ultimately the thrust H′ on the bottom bed joint surface *ab*, which must, therefore, exert an equal and opposite resistance H′ to meet it. The consequence is that the stress diagram representing the forces which are kept in equilibrium at H is P*efgh*′P of *Fig.* 1, the necessary modification being shown by dotted lines. It should be observed that the *line of resistance* of this system of loads is *fgh*, and that it does not coincide with the so-called line of active thrust for reasons which are explained in para. 52. The subject of the Stability of Structures of Uncemented Blockwork is fully dealt with in Chapter X.

The intermediate case is that in which the directions of the supporting pieces are all parallel to that of the remaining or *closing* piece AH; that is—

III.—The case of a parallel flanged girder or beam, including braced structures, such as the Warren and Whipple-Murphy girder, and beams of uniform strength with parallel flanges, the upper longitudinal fibres or flanges of which are subjected to compressive, and the lower to tensile strain only (*Plate* II., *Fig.* 7).

Cantilevers, the material of which is capable of resisting both compressive and tensile strain, belong to this class, including examples of cranes and balconies of wood or iron, and structures of cemented block-

work, such as isolated walls, &c. It should be noted that in these structures, which do not span an interval, the position of the fibres, subjected to tensile and compressive strain, is reversed with regard to the neutral surface (compared, that is, to their relative position in beams) those in tension being above, and those in compression below, it, relatively, that is, to the direction of the load. The analogy between the cantilever and beam, similar in form and similarly loaded, is fully dealt with in Chapter XIV., together with para. 91*a*.

19. It should be observed that in the case of the suspension cable and masonry arch the closing piece (stress 8, *Fig.* 1, *Plate* I.) is afforded by external agency—in the former case by the anchoring chain, in the latter by the abutments, and, therefore, in order to render these cases the equivalents of Case III., we must suppose imaginary and weightless pieces to take the place of the chain or abutment. In the suspension cable we suppose a weightless compression bar, in the masonry arch, a weightless tie-rod, to extend from A to H (*Plates* I. and II.), thus leaving in each case resultant resistances at the points of support, having the sum of their components resolved in the direction of the total resultant load together equal to that load (for the components resolved perpendicular to that direction cancel).

20. It will be seen, then, that, in order to ascertain whether the *first condition of equilibrium* is fulfilled by any given structure, it is sufficient to describe a polygon of external loads, similar to *abcdefg*, *Fig.* 1; close it by straight lines, as *gh*, *ha*, drawn parallel to the directions in which the resistances at the points of support are constrained to act; and then examine whether the structure is capable of affording the resistances so determined. The resistances at the points of support will, of course, unless constrained to act otherwise, always tend to act parallel to one another and to the direction of the total resultant load, as *ga*, *Fig.* 1.

21. In order to examine whether the *second condition of equilibrium* is fulfilled, a suitable pole, as P, *Fig.* 1, must be chosen, and a stress diagram and corresponding equilibrium polygon described as in *Figs.* 1, 2 and 3, and the effect on the structure of the application of such resultant stresses as the polygon indicates as necessary must be duly considered in a manner to be more fully described.

It will be observed that in the case of structures which do not span an interval, the material of which is only capable of resisting one kind

of strain, the pole P is taken *in* the polygon of external loads. For instance, in the simple illustration of para. 18, the pole P is taken at one extremity of force 5.

Further, if the material of the structure be capable of resisting both tensile and compressive strain, and the pieces designed to resist the two kinds of strain are parallel, then the pole is necessarily situated at an infinite distance. For instance, the vertical loads applied to a horizontally supported beam are resisted by horizontal stresses; horizontal loads applied to a vertical wall of cemented blockwork are resisted by vertical stresses.

22. The third condition of equilibrium has been fully dealt with under the head of Transverse Strain in Part II. of Vol. I. The graphic method of determining the values of bending moments and shearing forces will be discussed in the following pages, and its important bearing on the determination of lines of resistance pointed out.

23. It will be convenient to deal first with the representative frames and stress diagrams described in para. 18, and the chapters thus treating of the theory of the subject will form Part I. of this Volume. The application of such theory to the different classes of structures in actual use will be reserved for Part II. In doing this it will be convenient in Part I. to deal primarily with structures employed in spanning an interval, explaining at the same time the application of the principles to structures not so employed. In Part II. we shall reverse the process, and commence with the consideration of the more simple case of structures which are not employed in spanning an interval, passing then to structures that are so employed.

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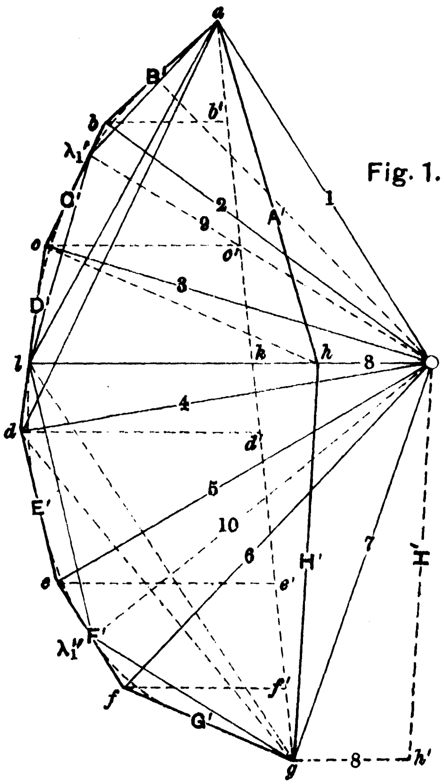


Fig. 1.

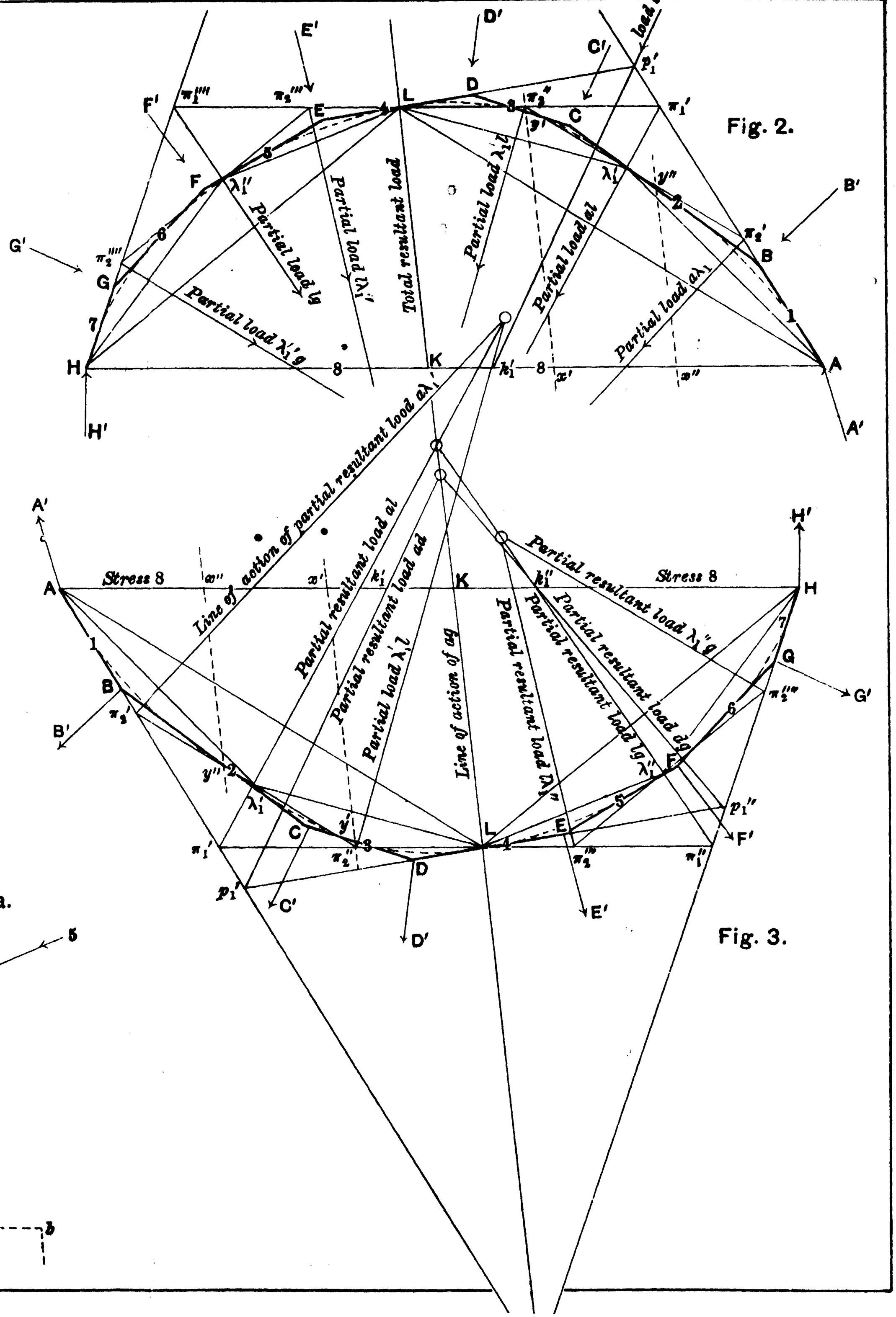


Fig. 2.

Fig. 3.

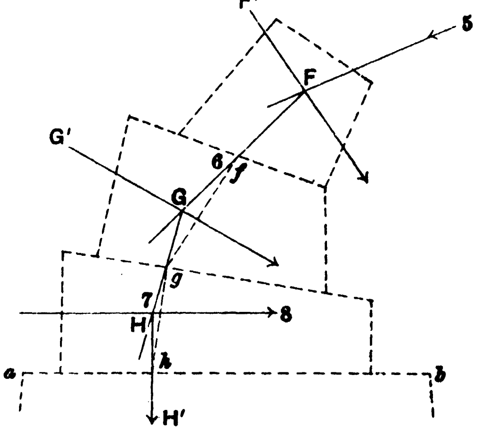
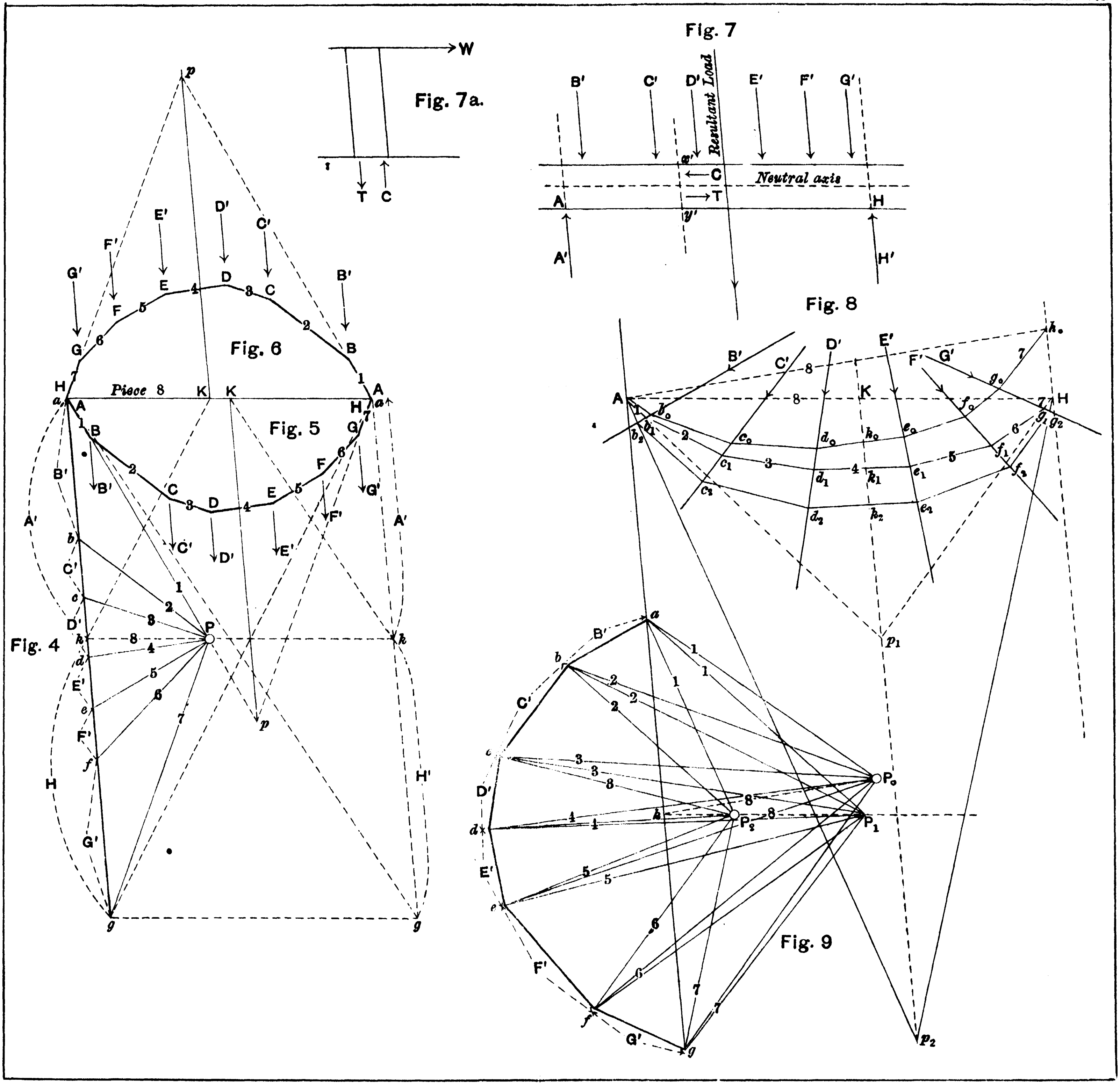


Fig. 2a.



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CHAPTER II.

ON THE MANNER IN WHICH THE THREE CLASSES OF STRUCTURES RESIST THE SHEARING AND BENDING FORCES.

24. An interval may be spanned in either of the three ways enumerated in para. 18 of the last Chapter, viz. :—

Case I., by structures, the pieces of which, immediately supporting the load or loads, are entirely subjected to tensile strain (*e.g.*, suspension structures), *vide Fig.* 3, *Plate* I.

Case II., by structures, whose supporting pieces are entirely thrown into compressive strain (*e.g.*, the arch), *Fig.* 2, *Plate* I.

Case III., by wooden beams, or parallel flanged girders (*Fig.* 7, *Plate* II.)

And it has been shown that structures which do not span an interval may be regarded as coming under Case II. or Case III., according as the material of which they are composed is capable of resisting compressive strain only, or both compressive *and* tensile strain simultaneously.

25. It has been shown that, in order to examine the stability of any one of these structures, the first thing to do is to reduce the system of loads, acting between the points of support and however distributed, to a system of detached loads, exactly equivalent in effect to that of the distributed loads. As all the structures, with exception of the cable freely suspended for bridging purposes, are either naturally stiff (*e.g.*, the arch), or artificially rendered so by bracing (*e.g.*, braced truss and girder), the lines along which the resultant applied loads act may be regarded as fixed in position relatively to the points of support.

26. The next thing to do is to describe an imaginary frame, with its angles lying on the lines of action of these resultant loads, and of such figure that, although unbraced, it is capable of balancing those loads. From the description of the method given in the previous Chapter it is evident that, supposing no further conditions requiring fulfilment were imposed, the number of frames capable of so balancing a given system of resultant applied loads would be unlimited. And it is, therefore, to the

third condition of equilibrium and to constructive requirements, such, for instance, as are imposed by the nature of the *joints* of the structure, that we must look for the necessary data to guide us in the selection of the particular frame, which will enable us to ascertain the true line along which the passive resistance of the material of the structure is actually exerted.

The frame and stress diagram together enable, in fact, the directions and magnitudes of a system of forces to be determined, which are the exact equivalents in effect of the external loads applied to the structure, the lines of action of which, however, are adapted to the particular form of the structure under consideration; and by simply supposing the directions of these forces to be reversed, the system may be regarded as either the active or passive equivalent of the applied loads. It will be observed that of all the sides of the frame that one only which represents the closing piece, numbered 8 in *Figs.* 2 and 3, represents a *passive* resistance actually afforded by the structure, for the line of resistance of the structure will, according to the definition given in para. 5, depend upon the position and nature of the *joints*. If, then, we regard the sides of the frame, other than the closing piece, as the active equivalents of the applied loads, the line of resistance of the structure will be determined by the points in which the joints are met by those sides. Thus, the stress diagram and frame, taken together, enable the stability and strength of the structure to be measured.

27. It is of the utmost importance, therefore, before proceeding further, that the student should thoroughly realize that the structure, however solid, and its system of applied loads, however distributed, must be reduced to two *skeleton* systems of forces, the one *active*, and embracing the equivalents of the applied loads, the other *passive*, and marking the lines along which the material of the structure must exert its required resultant resistance, in order to balance those applied loads. He must further realise that these skeleton systems *are* for stability measurements the equivalents of the actual systems; and he must also remember that, since the external forces are supposed to be applied *at the joints of the hypothetical frame only, the pieces of that frame can, themselves, be only subjected to direct longitudinal stress.* For it is thus only that they can yield the resultant compressive stress-line of the arch (*Fig.* 2) or the resultant tensile stress-line of the suspension structure (*Fig.* 3). Compare para. 52.

28. But, although these pieces are themselves only subjected to direct stress, it is yet evident, from the definition of Transverse Strain given in Chap. VI., Part II., Vol. I., that the structure, *as a whole*, is under Transverse Strain, because it is supported in the case of a beam at its extremities only, and in the case of a cantilever at one extremity only, and is, moreover, loaded between the two points or beyond the single point of support, so that the applied loads are only *indirectly* supported (*i.e.*, balanced by the fibres of the material of the structure). The consequence is that, if we consider the equilibrium of any portion of the structure lying to right or left of a plane, drawn parallel to the resultant load (perpendicularly to the plane of the paper) and dividing the structure, we shall find that it is kept in its place by the action of two equal and opposite couples, the one active and tending to make it revolve in one direction about some axis in the plane of its section, the other passive and tending to make it revolve in an opposite direction. The force of the active couple is the bending force, which is the resultant of all loads (including resistances at points of support) acting external to the portion of the structure considered, and is known as the *shearing force* and its moment as the *bending moment*. The elements of the opposite or resisting passive couple, *known as the moment of resistance*, are afforded entirely by material resistances at the section itself.

29. The general relation obtaining between these couples is thus expressed in para. 184 of Vol. I., viz., Cd' or $Td' = \mathfrak{M} = M$.

30. Now, for beams or cantilevers included under Case III., this resisting couple is obviously afforded by the resultant compressive and tensile fibrous resistances above and below the *neutral surface* (or longitudinal surface of the beam, the fibres of which are neither elongated nor compressed), multiplied by the distance between the centres of resistance of these two sets of fibres (*Plate* II., *Figs.* 7 and 7*a*), *vide* Vol. I., Chap. II.

31. And it is easy to show that for structures which span an interval and are included under Cases I. and II., the bending moment at any section is resisted in a similar manner, namely, by a couple, which is made up of the stress developed in the closing piece, and which is numbered stress 8 on *Plate* I., multiplied by the distance between its direction and the centre of stress of the piece which is cut by the section in question.

32. For it has been already pointed out that, in order to render suspension and arch structures the equivalents of the beams, included under Case III., it is necessary to suppose imaginary and weightless pieces substituted for the closing pieces of the equilibrium polygons (corresponding to piece 8 of the polygons shown in *Figs.* 2 and 3), such resistance being, in fact, afforded by external agency—in the suspension structure by the anchoring chain, in the arch, by the abutments. We thus obtain in each case closed frames. The triangular truss, bowstring girder, and curved beam of uniform strength are examples of Cases I. and II., in which the stress developed in the closing piece (which is numbered 8 in the diagrams) is afforded by a piece of the structure itself.

33. Consider, then, the forces acting at any section $x'y'$, *Figs.* 2 and 3, taken parallel to the direction of the resultant load ag, and cutting bar 3. The active forces to the left of the section in *Fig.* 3, and to the right of it in *Fig.* 2, are A$'$, B$'$, C$'$, which are equivalent to a single resultant whose magnitude is hc (*Fig.* 1), and whose line of action passes through the intersection of bars 3 and 8 and is parallel to hc (*Fig.* 1). Take moments about the centre of stress of bar 3. The active *bending moment* is the moment of hc; the passive *moment of resistance* is the moment of stress 8. These must, therefore, be equal and opposite if equilibrium obtain.

34. Moreover, since stress 3 may be resolved at the point where it meets the section into forces ch, hP, *Fig.* 1, which are respectively equal and opposite to hc and stress 8, we see the system is made up of a pair of equal and opposite *couples*, and we may substitute the word "couple" for the word "moment" in the above statement.

35. Since, then, stress 8 (the steadying stress) is constant in direction and magnitude for any particular frame and load distribution, the ordinates of the corresponding equilibrium polygon drawn from the line representing the closing piece 8, parallel to the resultant load to meet the polygon, become equivalent to a graphic representation, on no particular scale, of the values of the bending moments acting at the several sections of the structure, since the bending moment must, in all cases, be equal to the moment of resistance if equilibrium obtain. For instance, the relative values of the bending moments at sections $x'y'$, $x''y''$, *Figs.* 2 and 3, *Plate* I., are severally proportional to the lengths of the straight lines $x'y'$, $x''y''$, &c., being actually equal to those lengths severally multiplied by stress 8.

The graphic representation above described arises at once out of the general relation established in para. 91a, *q.v.*, which is applicable alike to beams and cantilevers.

36. In a similar way the values of the shearing forces at all such sections, being the resultants of external loads acting on either side of the section, may at once be obtained from the polygon of external loads, and their components, resolved in any one of two given directions, may be shown graphically by a polygon. Suppose, for instance, the shearing forces resolved parallel to the direction of the resultant load and to that of stress 8. Their several values are shown on *Fig.* 1. Thus, from joint A to joint B of frame (*i.e.*, throughout bar 1) the shearing force *on the right of the section* (para. 170, Vol. I.) is constant and equal to ak in the direction of the resultant load, and kh parallel to stress 8 (*vide Fig.* 1); throughout bar 2 it is equal to $b'k$, parallel to the resultant load, and $(bb' + kh)$ parallel to stress 8; throughout bar 3 it is equal to $c'k$ in the direction of the resultant load, and $(cc' + kh)$ parallel to stress 8, and so on; which lengths may be set off as ordinates from any given line of abscissæ, as AH of *Figs.* 2 and 3.

37. We can thus obtain a graphic representation both of the shearing forces and bending moments acting at all sections of the structure.

It may be remarked that in the case of flanged girders it is found that the effect of the bending moment at any section is resisted almost entirely by the material of the flanges; that of the shearing force by the material of the web, cut by the plane of the section (*vide* Vol. I., para. 186).

37a. Of structures which do not span an interval, the subject of the graphic representation of the shearing force and bending moment in cantilevers is fully dealt with in paras. 182 of Vol. I. and 91a of this Vol. And as to Structures of Uncemented Blockwork, since the simultaneous development of compressive and tensile strain in the material of the structure is of the essence of Transverse Strain, if the former condition be absent, the latter is so too. Structures of Uncemented Blockwork are, in fact, subjected to Strain of one kind only, and, therefore, according to fundamental mechanical laws, the lines of action of the forces acting on them must be confined to the structure; hence the absence of a closing stress in their stress diagrams, and the fact that the pole must be a point lying *in* the polygon of external loads.

CHAPTER III.

ON THE PROPERTIES OF THE EQUILIBRIUM POLYGON IN ITS RELATION TO LINES OF RESISTANCE.

38. IN the previous Chapter it was shown that a graphic representation of the value of the bending moment at any series of sections and for any given resultant load distribution may be always obtained by drawing a stress diagram and its corresponding equilibrium polygon; a similar representation of the corresponding shearing forces may be also obtained by setting off ordinates in the proper direction and of the proper length, these lengths being measured off the same stress diagram. The ordinates of the polygons can then be reduced to any convenient scale.

39. But the equilibrium polygon, besides affording the graphic representations above alluded to, also enables the magnitudes of the resistances at the points of support and the magnitudes and lines of action of any partial as well as of the total resultant load to be at once determined.

40. Before proceeding further, however, it should be observed that the application of the equilibrium polygon enables *two*, and *only two*, unknown quantities to be determined. Thus, the magnitude and direction of a single unknown force, or the magnitudes *or* directions of two forces whose directions *or* magnitudes are known, may be ascertained. For the open polygon whose sides represent all the known forces acting at the point, when taken in due order, can but be closed either by a single straight line, representing the direction and magnitude of a single unknown force, or by two straight lines whose directions *or* magnitudes must be known.

41. Considering, then, first the resistances at the points of support (*Plates* I. and II.), suppose the system of externally applied loads B′, C′, D′......G′ to remain constant, and also the direction AH of the

steadying resistance or stress 8; consider, in fact, the general case of an interval AH spanned by any braced or stiffened structure whatever and acted on by any given system of external loads, B′, C′, D′,......G′ whose magnitudes and lines of action (with reference to the points of support) are known.

If the resistances at the points of support A and H are constrained to lie in certain directions oblique to the resultant load, it is sufficient to describe the open polygon of external loads, *abc......g*, and close it by straight lines drawn in the required directions, when the magnitude of each resisting force on the scale of loads will be at once known (*Fig.* 1).

If, however, the resistances be not constrained to act in certain directions, clearly they will, if possible, take a direction which is parallel to that of the total resultant load.

Draw the lines of action (*Plate* II., *Fig.* 8) of the applied forces B′, C′......F′, G′, which are supposed to act on the interval AH, between the actual points of support A and H of the structure.

Draw the polygon of external loads *abcd......g* (*Fig.* 9) as before and join *ag*; then *ag* represents the total resultant applied load acting on the structure, and *ga* the sum of the resistances at the points of support A and H.

Join AH (*Fig.* 8); then AH represents the direction of the steadying stress; through A and H draw straight lines parallel to *ag*, to represent the limits within which the frames are to be drawn.

Take *any* pole P_0 and describe the corresponding stress diagram P_0 *abcd......g* P_0 (*Plate* II., *Fig.* 9) and frame diagram (*Fig.* 8) Ab_0 c_0 d_0......g_0 h_0; join Ah_0; then Ah_0 represents the direction of stress 8 for this particular frame. Through P_0 draw P_0k (shown by a dotted line in *Fig.* 9) parallel to Ah_0 meeting *ag* in *k*. Then the straight lines *ak*, *kg* represent the resistances at the points of support A and h_0 respectively. But by an elementary statical theorem, a force may be supposed to act at any point in its line of action, hence *ak*, *kg* represent the resistances at the points of support A and H, as well as A and h_0.

42. In order to describe a frame whose extremities actually lie in the points A and H proceed as follows :—

Through *k* draw a straight line kP_2P_1 parallel to AH, to represent

the locus of poles, which will yield frame diagrams whose steadying stress 8 lies in the straight line joining A and H.

Take any poles, as P_1, P_2, and describe the corresponding stress and frame diagrams. In the first case we obtain the frame $Ab_1c_1d_1 \ldots g_1$ H; in the latter the frame $Ab_2c_2d_2 \ldots g_2$ H, the extremities of each of which terminate in the points A and H.

43. Moreover, by producing the directions of the extreme bars or links 1 and 7 to meet in a point (p_1 or p_2), and drawing through that point a straight line parallel to the direction of the total resultant load ag, we obtain the line of action of that total resultant load (as has been already pointed out) which divides the interval AH at the point K into parts such that $\frac{\text{AK}}{\text{KH}} = \frac{\text{resistance at H}}{\text{resistance at A}} = \frac{gk}{ka}$.

44. Having drawn one frame diagram corresponding to a frame whose extreme angles lie in the required points of support A and H,—having, that is, drawn a graphic representation of the values of the bending moments at the several sections of the structure,—we can at once describe another frame having any required frame-depth by taking the new pole P_2 at such a distance from ag, that the new value of stress 8 shall be to the old value in the inverse ratio of the corresponding frame-depths; that is to say, so that $\frac{P_1k}{P_2k} = \frac{Kk_2}{Kk_1}$

whereby the value of the bending moment at any section of the structure is maintained constant.

45. The previous investigation evidently holds good whether the applied loads be parallel or oblique to one another, provided only that their directions be maintained constant, that is, provided that the structure be sufficiently stiff to resist deformation.

46. The case of the uniform, flexible, heavy cable, acted on by any system of loads, whose magnitudes and directions only, with reference to the points of support, are known and applied at known distances along the rope, is of rare occurrence.

47. Now the line of action and the magnitude of the resultant of any partial loading may be determined in exactly the same way as the line of action and magnitude of the total resultant load were determined, viz., by closing the open polygon formed by the loads in question, and thus determining the direction and magnitude of the resultant load, and then by producing the directions of the lines of action of the extreme

bars or links supporting the system of loads in question to meet in a point, and drawing through that point a straight line parallel to the partial resultant load as previously determined on the stress diagram.

For instance, loads B′, C′, D′, are supported by bars 1 to 4 (*Fig.* 1). Join *ad* in the stress diagram; then *ad* represents the resultant of B′, C′, D′, both in direction and magnitude. Produce bars 1 and 4 of the frame (*Fig.* 2 or 3) to meet in p_1', and through p_1' draw $p_1'k_1'$ parallel to *ad*. $p_1'k_1'$ represents the line of action and *ad* the magnitude of the resultant of the partial loading considered.

48. In a similar manner the line of action $k_1''p_1''$ of *dg*, the resultant of the partial loading E′, F′, G′, may be determined, and it is evident that the straight lines $k_1'p_1'$ and $k_1''p_1''$ will meet the line of action of the total resultant load KL in one and the same point. And so on for any other partial resultant loads.

49. We are now in a position to enunciate the principle of projection by parallel rays of such frames as those shown in *Figs.* 2, 3, 5, 6 and 8. The principle is thus stated on p. 106 of Rankine's Applied Mechanics (3rd Edition):—"If a frame, whose lines of resistance constitute a given figure, be balanced under a system of external forces represented by a given system of lines, then will a frame whose lines of resistance constitute a figure which is a parallel projection of the original figure, be balanced under a system of forces represented by the corresponding parallel projection of the given system of lines; and the lines representing the stresses along the bars of the new frame will be the corresponding parallel projections of the lines representing the stresses along the bars of the original frame," which "theorem enables the conditions of equilibrium of any unsymmetrical frame, which happens to be a parallel projection of a symmetrical frame, to be deduced from the conditions of equilibrium of the symmetrical frame"; (for instance, the state of equilibrium of the skew arch may be deduced by projection from that of the corresponding right arch).

50. If, then, the system of loads be parallel, as in *Figs.* 5 and 6, *Plate* II., and the plane of the frame be supposed to be revolved about the line of action of the total resultant load, and thence projected by parallel rays on to the plane of the paper, it is evident that the projection so obtained will represent a frame loaded exactly similarly to the original frame, excepting only that the parallel loads will be nearer

together and the span shorter compared to the frame depth; in other words, the scale of abscissæ-measurements (along AH) will be diminished, while that of ordinate measurements (parallel to Kp) is retained, *i.e.*, relatively increased.

If further the corresponding stress diagram (*Fig.* 4) be revolved about ag in exactly the same way and similarly projected on to the plane of the paper, we shall evidently obtain a projection of the stress diagram, representing a system of forces which would severally balance if applied similarly at the corresponding points of the new frame, as the corresponding original forces were applied at the corresponding points of the original frame.

In a similar way the frame might be revolved about the axis of abscissæ AH (thereby diminishing the scale of ordinates while retaining that of abscissæ) with exactly similar results, provided only that the corresponding stress diagram be supposed to be similarly revolved about the line Pk, corresponding to the direction of the steadying stress 8.

51. If, however, the operation above described be applied to the case of oblique loading (*Figs.* 2 and 3), it is evident that the directions of the oblique loads will become altered by projection relatively to the straight line joining the points of support, and that otherwise the principle is quite applicable. Hence we deduce the following statement :—The projection of a stress diagram by parallel rays will represent a system of forces which will balance if applied to the angles of the corresponding frame similarly projected from the original frame, but the directions of the oblique loads, if any, will be altered relatively to the line joining the points of support. Hence, *in the case of parallel loading, the several equilibrium polygons possible may be regarded as all really representing one and the same polygon, only plotted differently.* They may, in fact, be all regarded as shadows, or parallel projections, of some ideal polygon of bending moments, existing in some plane which is inclined to that of the paper, and which is projected on to it by parallel rays, sloping at a different angle to it for each new projection.

CHAPTER IV.

ON LINES OF RESISTANCE AND BEAMS OF UNIFORM STRENGTH.

52. It has been already pointed out that the line of resistance of a structure is determined by the line, straight, angular, or curved, which joins the points in which the joints are cut by the sides or links of the equilibrium polygon. Now, although, in order to measure merely the stability of a structure, it is sufficient to take into account the resultants only of the loads acting between the joints, yet the polygons so obtained do not represent the actual lines of action in the most usual cases, those of uniformly loaded structures. When the distribution of the load is taken into account, the polygons obtained in the manner already described merge into curves, and the sides of those polygons become tangents to the latter, and the lines of action and re-action become identical, the equilibrium polygon now merging into the line or *curve* of resistance.

Thus, the polygons of *Figs.* 2 and 3 become the curves indicated by dotted lines. The dotted curve of *Fig.* 3 exhibits the form which a perfectly flexible, weightless, and inextensible string would assume, whose length is equal to that of the curve, were it acted on by uniformly distributed loads whose resultants between the points of contact A, 2, 3,....H, are represented in direction, magnitude, and position by the forces B′C′D′ . G′ of *Fig* 1 ; while the dotted curve of *Fig.* 2 represents a line of resistance such as would occur in the masonry arch of Case II., or the resultant line of all the resistances uniformly distributed among the several pieces of the braced structures, included in that case, under similar conditions.

53. The analogy between the tension line of resistance of Case I. and the compression line of Case II. is thus at once apparent. "Conceive a cord or chain to be exactly inverted so that the load applied to it, unchanged in direction, amount, and distribution, shall act inwards

instead of outwards; suppose, further, that the cord or chain is in some manner stiffened, so as to enable it to preserve its figure, and to resist a thrust; it then becomes what is known as a *linear arch*, or *equilibrated rib;* and for the pull at each point of the original cord is now substituted an exactly equal *thrust* along the rib at the corresponding point" (Rankine's Applied Mechanics, p. 182).

54. The curve of resistance will, in any case, be a *catenary*, which may be defined generally as the figure which a uniform flexible rope or chain assumes when acted on by any system of loads whatsoever. (Thomson and Tait's Elements of Natural Philosophy, Part II., para. 157, *et seq.*) If the chain be supposed weightless, and the load distributed uniformly along a horizontal line, then the catenary becomes a parabola which is the figure a suspension chain would assume, *were it weightless*, and the bridge load distributed uniformly and horizontally; it is also the figure of the curve of bending moments of any horizontal beam or girder uniformly loaded with its own weight. If a chain be supposed to be heavy and to be loaded with its own weight only, the curve which it assumes becomes what is known as the *common catenary.* These are all special forms of the genus "*catenary.*"

55. The polygon of external loads then (*Fig.* 1) will be more truly represented by the dotted curve drawn tangential to the sides of the polygon *abcdefg*, and the line of resistance (*Figs.* 2 and 3) by the dotted curves drawn tangential to the sides of the equilibrium polygons. From the nature of the latter curves each vector, such as P*c*, P*b*, &c., drawn from the pole P to the former curve *abcdefg*, becomes parallel to a tangent to the latter at a certain point which may be determined in a manner to be explained.

56. First, suppose a plane to be drawn perpendicular to that of the frame so as to contain the line of action of the total resultant load KL (*Figs.* 2 and 3). The frame is thus divided into two parts, and we may deal with each separately.

Join *al*, *lg* (*Fig.* 1). Then *al* represents the total resultant load acting on the partial frame AL (*Figs.* 2 and 3), and *lg* (*Fig.* 1) that on the partial frame LH.

Through L (*Figs.* 2 and 3) draw a straight line parallel to stress 8 (*i.e.*, to AH), and produce its direction to meet that of bar 1 in π_1' and bar 7 in π_1''.

If through π_1' a straight line be drawn parallel to *al*, and through π_1'', one parallel to *lg*, obviously the former will represent the line of action of the load on the partial frame AL and the latter on LH.

Let the former cut the equilibrium curve in λ_1' and the latter in λ_1'' (*Figs.* 2 and 3).

Now resolve the forces *al* and *lg* acting at the points π_1' and π_1'' respectively parallel to KL (the direction of the total resultant load) and to stress 8 (*i.e.*, to AH).

Obviously the former pair of forces (*Fig.* 1) are represented by *ak* and *kg* respectively, and the latter by *kl* and *lk* respectively.

But if the resistances A′ and H′ at the points of support A and H be likewise resolved in these same directions, we have already seen that the components parallel to the direction of the total resultant load are *ka* and *gk* respectively.

57. Hence, *if a plane, containing the line of action of the total resultant load acting on a frame, be supposed to intersect that frame perpendicularly to its plane, it will divide the structure into two parts, such that the resultant partial loads on each when resolved parallel to the direction of the total resultant load acting on the structure, are severally equal to the parallel resistance at the corresponding point of support;* in other words, the component of the applied load, resolved in the direction of the total resultant load, acting on the left partial frame is equal to the component of the resistance at the left point of support resolved in the same direction; and the similarly resolved part of the active forces on the right partial frame is equal to the similarly resolved part of the right resistance at the right point of support.

58. Moreover, since the vectors drawn from P to the curve *abcd*... ..*g* (*Fig.* 1) are all tangents to the curves A 2 3....6 H (*Figs.* 2 and 3), it follows that the tangent at L is parallel to AH and those at all other points inclined to it.

Hence, *the line of action of the total resultant load contains the greatest ordinate* (KL) *of the equilibrium curve, or curve of bending moments, or curve of resistance, measured in that direction, and the plane containing that line of action, therefore, contains the section of maximum bending moment.*

59. Now by joining AL, LH (*Figs.* 2 and 3), we obtain two partial frames ABCDL and LEFGH, which may be dealt with in exactly the

same manner as was the complete frame, al being the total resultant load on the former and lg on the latter.

Through λ_1' (*Figs.* 2 and 3) draw a straight line parallel to AL, and through λ_1'' one parallel to LH. These are tangents to the curve.

By joining $A\lambda_1'$, $\lambda_1'L$, and $L\lambda_1''$, λ_1'' H, we further divide the frame into four partial frames, which may in like manner be dealt with separately.

Draw the vectors $P\lambda_1'$, $P\lambda_1''$ (*Fig.* 1) parallel to AL, LH respectively, and join $a\lambda_1'$, $\lambda_1'l$, $l\lambda_1''$, $\lambda_1''g$ (*Fig.* 1), and let the tangent at λ_1' (*Figs.* 2 and 3) produced meet stress 1 in π_2' and the tangent at L in π_2'', and let the tangent at λ_1'' meet that at L in π_2''' and stress 7 in π_2'''' (*Figs.* 2 and 3).

Then, the resultant load $a\lambda_1'$ (*Fig.* 1) acts through π_2' (*Figs.* 2 and 3).

"	"	$\lambda_1'l$	"	"	π_2''
"	"	$l\lambda_1''$	"	"	π_2'''
"	"	$\lambda_1''g$	"	"	π_2''''

and tangents may be drawn, as before, at the points where the line of loads drawn through π_2', π_2'', π_2''', π_2'''' meet the curve, and so on.

60. In this way a curve of resistance may be interpolated between the sides of an equilibrium polygon.

61. It will be noticed that any equilibrium curve, described for the given system of loads, enables a beam of uniform strength to be designed, which shall be capable of sustaining such system of loads. For the ratios which the ordinates at successive sections of any equilibrium curve whatever, so described, bear to one another—no matter whether its axis of abscissæ be horizontal or not,—represent the ratios which the bending moments at these sections bear to one another, and the effect of the bending moment is resisted by the flanges. If, then, the beam is to have parallel flanges, the distance between their centres of stress (that is, the length of arm of the resisting couple) must be maintained constant, and consequently, if the width of the flanges is to remain constant also, their thicknesses must vary, and if their thickness is to remain constant as well as their distance apart, the *widths* must vary, and the curve of bending moments (*i.e.*, the equilibrium curve) will enable the proper variation to be ascertained. If, on the other hand, the width as well as the thickness of the flanges are to be maintained constant, then the shape of the web

PLATE III.

Fig. 10

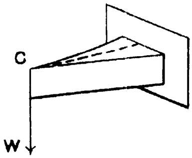


Fig. 12

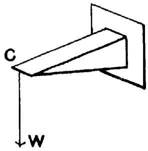


Fig. 11

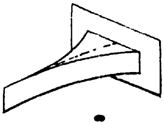


Fig. 13

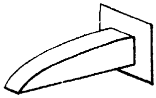


Fig. 14

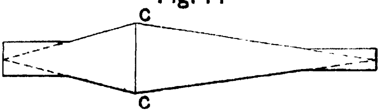


Fig. 15

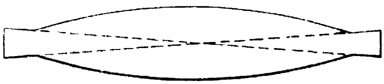


Fig. 16

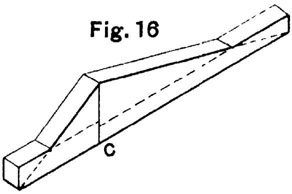


Fig. 17

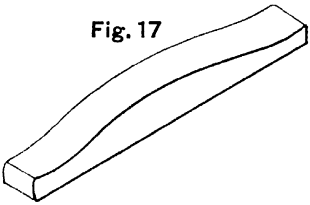
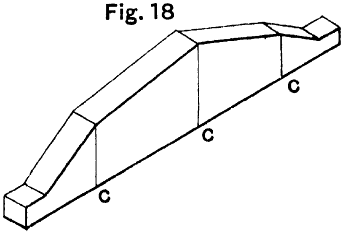


Fig. 18



must evidently be altered in conformity with that of the equilibrium curve, in order to afford the necessary variation in length of the arm of the resisting couple, the forc of which, corresponding in this case to constant flange-cross section, is also constant. Hence the ordinates of the equilibrium curve, when reduced to a proper scale and plotted from a proper axis of co-ordinates, actually, in this case, represent the proper lengths of the arms of the corresponding couples of resistance, and so the actual form of the beam.

Figs. 10 to 18, *Plate* III., illustrate this; *Figs.* 10, 12, 14, 16, and 18 being designed to carry isolated loads hung at points C, and *Figs.* 11, 13, 15, and 17, uniformly distributed loads. *Figs.* 14 and 15 are plans of such beams, the depths being constant; the other figures are isometrical projections. They are all hypothetical cases, intended to illustrate the principle only.

CHAPTER V.

JOINTS IN THEIR RELATION TO LINES OF RESISTANCE AND LINES OF LEAST RESISTANCE.

62. SINCE the compressive state of strain is one of unstable equilibrium, all the structures included under Case II. are necessarily either braced or stiff. Those included under Case I., with the exception of the freely suspended cable, are likewise also braced. For, if a suspension structure be not braced, that is, be flexible, the lines of action of the applied loads are liable to change, relatively to the points of suspension, if the magnitudes of the loads be changed, whereas if the structure be braced, the lines of action of the loads are maintained constant however much their magnitudes (within proper limits) may be changed.

63. Now, it has been already stated that, leaving out of consideration the case of the solid wooden beam (which is fully dealt with in Part II., Vol. I.), the joints of all the structures, with the exception of the masonry arch, are regarded as being of the first class, those of the masonry arch as of the second. Moreover, braced structures, such as the stiffened suspension bridge, triangular truss, open web girder, &c., are regarded as having the actual pieces of which they are composed freely jointed together, and as being *loaded at the joints only.*

We shall, therefore, leave the theoretical consideration of these structures at this point, because the positions of at least three joints are, as a rule, known, viz., those at two points of support and at some intermediate point, generally at the middle of the span, so that the required equilibrium curve can, as a rule, be at once described since three points on it are known; and we shall, therefore, pass on forthwith to the consideration of lines of *least* resistance.

64. There is this difference between determining the true, or *actual*, curve of resistance of a structure whose joints are of the first class and that of one whose joints are of the second, that whereas in the

former certain conditions (as of loading, figure of structure, &c.,) being given, only one single curve can be drawn representing a line of resistance which is capable of balancing the applied loads (because the centre of resistance of each joint being at its middle point is fixed in relation to that joint, and the direction of the line is therefore determined); in the latter, on the contrary, since the nature of the joints admits of a certain area over which their centres of resistance may occur, several curves can, as a rule, be drawn, *each* of which, while falling within the prescribed limits of resistance area, represents a line of resistance capable of balancing the given applied loads. Now of all these possible curves, the *true* one—that is, *the* curved line along which resistance actually does take place—is that one along which the applied loads can be balanced by the material of which the structure is composed *with the least effort possible;* and this fact depends on the general physical law expressed in the following Theorem, known as the "*Principle of Least Resistance*" and first stated by Mr. Moseley, viz.:—

65. "If the forces which balance each other in or upon a given body or structure be distinguished into two systems, called respectively *active* and *passive*, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure.

"For the passive forces being caused by the application of the active forces to the body or structure will not increase after the active forces have been balanced by them, and will, therefore, not increase beyond the least amount capable of balancing the active forces."*

66. The question of the limits within which resistance at joints of the second class is possible is purely one of *strength*, and will vary according to the nature of the material of which the structure is composed.

67. In any case, however, as has been already pointed out, the lengths of the several ordinates of the resistance curve are proportional to the magnitudes of the resisting couples required to balance the bending moments at the several corresponding sections; and hence, for any particular bending moment, the longer the ordinate, the smaller the corresponding resisting force required to form the necessary couple; and, therefore, of all the curves possible under given conditions, that one will

* Rankine's Applied Mechanics, 3rd Edition, p. 215.

represent the curve of *least* resistance which, while falling within the prescribed limits of joint areas, is *the most concave of all with regard to the axis of abscissæ.*

68. Now in the general case of a masonry arch subjected to the action of oblique loads, since the angles of the several equilibrium polygons possible under given conditions must all lie on the lines of action of the corresponding loads, if the latter be all inclined at different angles to the direction of the total resultant load, the several values of the bending moment, measured at any section whose plane is parallel to that direction, will differ with different polygons, for they will vary according to the several positions of the angles of the polygons, that is, according to the manner in which the arch ring can resist the several applied loads at different points along their lines of action. Hence, the several polygons and corresponding curves of resistance must, in this case, all be different, although each will represent for its own particular loading the law of variation of the bending moment.

69. But in the case of a masonry arch loaded with a system of parallel loads, the value of the bending moment at any section whose plane is parallel to the direction of those loads is (as has been already pointed out) constant, and hence the law of variation of the bending moment is constant also for all equilibrium curves possible.

In this case, therefore, however flat or curved the lines of resistance may be (which is simply due to varying the scale to which the ordinates are plotted, while retaining that of the abscissæ constant), they really all, *cæteris paribus*, represent but one and the same curve, and may, as in the case of the equilibrium polygon, be regarded as *shadows, or parallel projections, of some one ideal curve of bending moments existing in a plane which is inclined to that of the arch section under consideration and projected on to the plane of the paper by parallel rays.* In this case, therefore, having drawn one equilibrium curve, in the manner explained, in order to ascertain whether its figure will lie within the two curves representing the available area of material resistance offered by the arch ring, it is open to us *either to make a projection of the equilibrium curve, or of the limiting curves, or of both,* if necessary, and this often facilitates the solution of the problem.

70. Further, in any example, whether of Case I. or II., provided the system of external loads remain constant, it has been already shown

that at that section which so divides the structure that the resultant partial loads on either side of it, measured parallel to the direction of the total resultant load (the other component being measured parallel to the axis of abscissæ), are respectively equal to the parallel resistances at the extreme sections, (that is, at the points of support) the direction of the line of resistance is parallel to the axis of abscissæ from which the ordinates of the curve are plotted, for at this section the value of the bending moment is at a maximum, and, consequently, that of the moment of resistance is at a maximum also. Hence, in the case of masonry arches, since the curve of resistance at this section will be more concave with regard to the plane passing through the lines of springing of the arch than the curve of the arch ring, and since the former must never pass outside the limits of joint-areas within which resistance is possible (which limits are bounded by curved surfaces parallel to those of the arch ring), it follows that at this section of greatest bending moment the curve of resistance will occupy its highest *allowable* position when it touches the curve representing the upper limit of resistance area.

71. Also a line of resistance can intersect a limiting ring curve at two points only, viz., at those two in which the latter meets the abutments, and at all other points can only touch it; and it may, moreover, never pass below the tangent to the lower limiting curve, or above that to the upper, at the abutments; consequently the directions of these tangents determine the limiting directions of resistance at these points. The resistance curve, therefore, will always touch the upper limiting ring curve in one point at least, and may either cut the lower limiting curve at the abutments, or touch it at a point or points higher up.

CHAPTER VI.

TO DRAW THE LINE OF LEAST RESISTANCE IN A GIVEN ARCH RING FOR A GIVEN SYSTEM OF EXTERNAL LOADS.

72. In order, then, to draw the line of least resistance in a given arch ring for a given system of applied loads, it is necessary to make the following four determinations, which become simple in the most common case of a symmetrical arch, symmetrically and vertically loaded.—

Step I.—To determine the section of greatest bending moment of the given arch ring under the given conditions.

Step II.—To determine whether the curve of least resistance will touch the upper limiting ring curve at the point in which it is intersected by that section, and if not, at what other point or points.

Step III.—To determine whether the said resistance curve will cut the lower limiting ring curve at the abutments, or touch it at some point or points further up, and if so, where.

Step IV.—To describe the resistance curve accordingly.

73. The following three cases will be found to include all those of ordinary occurrence, viz. —

Case A.—Where the loading is symmetrically situated with regard to the arch ring and vertical.

Case B.—Where the loading is unsymmetrically situated with regard to the arch ring and vertical.

Case C.—Where the vertical loading is symmetrically or unsymmetrically situated, and the arch is subjected, in addition, to the pressure of earth, the direction of which may be either horizontal or inclined.

This last is the most general case and includes examples of deep tunnels and underground cellars.

It will be noticed that Cases A and B are examples of parallel loading, while Case C is an example of oblique loading.

Determination of Steps in the different Cases.

74. *Step I. in all Cases.*—If Step I. is not self-evident, it is at once determined by means of *any* stress diagram and the corresponding equilibrium polygon for the given system of loads in the manner already explained.

75. *Steps II. and III. for Cases A and B.*—Since the structure is stiff the lines of action of the several loads remain unchanged, and since in the case of parallel loading the law of variation of the bending moment from section to section is constant, if one graphic representation of that law be determined, others may be described by projection. For regarding the equilibrium polygon already drawn as itself a projection of some ideal polygon of bending moments existing in a plane inclined to that of the arch-section under consideration (*i.e.*, to the plane of the paper), and intersecting it along a fixed straight line, the plane of the ideal polygon may be conceived so to alter its inclination with regard to the plane of the paper by revolution round that straight line, or the projecting lines or parallel rays so to alter their inclinations with regard to the plane of the paper, that the resulting projection of the curve of bending moments may fall entirely within the prescribed limits, and be as concave as possible with regard to the axis of abscissæ. The principles on which such a projection may be made are as follows:—

I. The sides of the polygon already drawn, which are tangents to the corresponding curve, and the similar sides of any and all its projections, if produced, will meet the axis of revolution or line of intersection of the two planes already referred to in one and the same point. Hence, since in the case of a stiff structure the lines of action of the several loads are fixed, if the figure of one projection and the position of the line of intersection of the planes be known, a new projection can be at once described, if only the direction of one side be known.

For, it is shown in para. 91, *q.v.*, that *for parallel loading, the length intercepted on any straight line, drawn parallel to the direction of the loads, by any two sides of the equilibrium polygon produced, is proportional to the moment about that straight line of the load which is balanced by those two sides.* This being true for each load, and the structure being stiff, and the values of the moments therefore constant for all the equilibrium polygons that can be described for the given system, the intercepts on any given straight line are constant also.

II. Since a re-projecting of the imaginary curve is equivalent to altering the scale to which its ordinates (or abscissæ) are plotted while retaining that of its abscissæ (or ordinates) constant, if the figure of one projection be given, and the new value of any one ordinate (or abscissa) be known, the new values of all the other ordinates (or abscissæ) may be determined by simple proportion.

The relations which the properties of the circle bear to those of the several conic sections, which may be described by parallel projection from it, afford a familiar illustration of the properties above described.

76. Step II. must be determined by inspection as follows :—

In Case A, the line of least resistance will, *as a rule*, touch the upper limiting ring curve at the section of greatest bending moment, and the common tangent at that point, therefore, become the axis of revolution, or common intersection of the two planes above referred to. Step III. will determine itself during the process of projection. An Example is given in Section III., Part II.

In Case B, Steps II. and III. are determined by comparing the form of the equilibrium curve with those of the limiting curves of the arch-ring; and in doing this the tangents to the former (or sides of the corresponding polygon), which require especial attention, are those in the neighbourhood of which the direction of the curve changes most suddenly. These may be called the *critical sides*. The comparison is best effected by projecting the ring curves on to the equilibrium polygon as follows :—

Using the resistance curve already drawn and plotting from its axis of abscissæ, or closing side (stress 8 of *Plates* I. and II.), make the ordinate of the upper limiting curve at the section of greatest bending moment equal to that of the resistance curve at that section, and proportionately alter the lengths of *all* the other ordinates of both limiting curves. The resistance curve will then cut the upper limiting curve thus transformed at the section of greatest bending moment. In other words, re-plot the limiting curves from the axis of abscissæ of the resistance curve, altering the scale of ordinates in the ratio of the length of the ordinate of the resistance curve at the section of greatest bending moment to that of the original upper limiting curve at the same section. The form of the resistance curve can then be readily compared with that of the limiting curves, and the point or points of probable contact ascertained.

Step II. having been thus determined by inspection, a tangent is drawn to the unprojected plotting of the upper limiting curve at its point of contact with the equilibruim curve (or line of least resistance), and the polygon, already drawn, or its corresponding curve, is projected in the manner already described by revolution round this common intersection of the planes, so as to fulfil, as far as possible, the required conditions. As a rule, a second projection will be necessary, and this must be made round the ordinate (produced) at the said point of contact. Step III. will, as before, determine itself during the process of projection. An Example of Case B is given in Section III., Part II.

77. *Steps II. and III. for Case C.*—Since in this case the equilibrium curves cannot be projected, the line of least resistance must be determined by trial, the limits of pole-area being first determined, and resistance curves described until a suitable one be obtained.

78. The method of finding the limits within which all possible poles lie is best explained by an example. Suppose a given arch-ring to be subjected to the action of any system of oblique loads, such as that shown in *Plate* IV. With any pole P draw a stress diagram and its corresponding equilibrium curve. Let p be the point in which the straight line Aa, representing the total resultant load, is cut by the closing vector P$p\pi'$. Aa is divided at p proportionately to the resistances (parallel to Aa) at the points of support A and E. Since the system of loading is constant, the closing vector of all stress diagrams corresponding to equilibrium curves which balance the system must pass through p. Hence all poles must lie within the straight lines x_1p and x_2p, which are drawn through p parallel to the straight lines a'E and Ae', which join diagonally the points A, a', e', and E, being the extreme corners of the limiting resistance area of the arch-ring.

79. The limits of pole-area in the other direction may be determined sufficiently accurately as follows :—It has been already shown that the tangent to the equilibrium curve at the section of greatest bending moment is parallel to the axis of abscissæ of the curve. Let this tangent (*Plate* IV.) meet $S_1'g_1$, the line of action of the resultant Aπ of all the loading which acts to the left of the section of greatest bending moment gg_1, in the point S_1'. Suppose the resultant load Aπ to be resolved at S_1' into Ap (*Fig.* 20) parallel to Aa, and $p\pi$ parallel to the axis of abscissæ. The latter is counteracted at S_1' by the stress πP

acting along the tangent at S. There remain at S_1' the two forces Ap and pP.

Now the bending moment at G_1 (*Fig.* 19), the point in which the line of action of the total resultant load meets AE, is equal to the moment of pA at A, less that of Ap at S_1'; and the moment of resistance at G_1 is equal to the moment of pP at S.

If through S_1' the straight line $S_1'\sigma_1 s_1$, be drawn parallel to Aa, the bending moment at G_1 will be graphically represented by the area $A\sigma_1$, and if $\sigma_1 s_1$ be taken equal to G_1S, and the points s_1 and p be joined and produced to meet a straight line drawn through A parallel to Pp in y_1 and the parallelogram $y_1 s_1$ completed, then will the complement py_1' be equal to the complement $A\sigma_1$, and the former will represent the moment of resistance, so that Ay_1 becomes the polar distance corresponding to the ordinate G_1S.

Hence (*vide Plate* IV.), in order to find the polar distance corresponding to the point S_1' on the line of action $S_1'g_1$ of the partial load $A\pi$, draw through S_1' a straight line $S_1's_1$ parallel to Aa, making $\sigma_1 s_1$ equal to the ordinate GS of the upper limiting curve at the section of greatest bending moment, and complete the parallelogram $y_1 s_1$, as already explained.

The greatest possible ordinate of any equilibrium curve whatever of the given system is G_1S, G_1 being the point in which the straight line AE meets the line of action of the total resultant load gg_1, and S that in which the upper limiting ring curve meets it.

The least possible ordinate is G_2S_2, G_2 being the point in which the straight line $a'e'$ meets gg_1, and S_2 that in which the lower limiting ring curve meets it.

Since the most *concave* equilibrium curve possible is sought for, the former pole-locus, corresponding, that is, to the greatest possible ordinate G_1S, only need be drawn. (Both are shown in *Plate* IV.)

The required limits of pole area are thus represented by the figure $x_1x_2x_4x_3$, and the pole, corresponding to the line of *least* resistance, will be as near the limit x_3x_4 as possible.

80. If a pole be not readily found which yields an equilibrium curve fulfilling the required conditions, then the *critical* side or sides of the polygon already drawn must be determined by inspection in the manner already explained, and the lines of action of the resultants of the loads

balanced by it or them, on either side drawn in. A point must then be chosen in each line of action such that the straight line joining them, which gives the direction of the new critical side, together with the extreme sides of the corresponding polygon may yield a curve fulfilling the required conditions—remembering always that the two extreme sides of all polygons must meet in a point in the line of action of the total resultant load.

81. If the sides of the polygon be numerous, it should not be described by commencing at one extreme side, and then drawing the other sides successively, because the errors of drawing are cumulative, and slight ones must necessarily have already occurred in determining the lines of action of the resultant loads. Commencement should, therefore, be made at four points, viz., at each extreme side, and at each extremity of the critical side.

82. When drawing equilibrium polygons care must always be taken not to draw *reversed* polygons, but to see that the forces measured at the different points are considered in their proper order and direction. The greatest accuracy of drawing is always necessary.

The above description will be best understood by help of the following example :—

General Case of the Arch.

83. A given arch ring is loaded with any given system of loads, such as that shown in *Plate* IV.; required to draw the curve of least resistance.

Let the given arch ring be semi-circular and meet the abutments at points B′, B″ (*Plate* IV.) The polygon AA′B′C′D′E′F′G′a (*Fig.* 20) represents the load, which is borne by it, and resistance is only possible within the area indicated by the thick lines of the arch ring (*Fig.* 19). Required to examine if it be possible to draw a line of least resistance within these limits.

Join Aa (*Fig.* 20): then Aa represents the total resultant load both in direction and magnitude.

Conceive planes, at right angles to the plane of the paper and parallel to Aa, to pass through the points B′, B″, in which the arch ring meets the abutments, and let them intersect the limiting arch rings in A, a' and E, e' respectively. The limiting area of resistance is denoted by the semi-annulus Aa'Se'ES_2A, and within these limits the required line of least resistance must lie.

In order to find the line of action of the total resultant load, take any convenient pole P, and describe a stress diagram and the corresponding equilibrium polygon (as indicated by thin lines, *Figs.* 19 and 20), taking care that all its sides be numbered similarly to the corresponding stresses in the stress diagram; that all the forces acting on the arch ring be included*; and that its extreme angles lie along the straight lines passing through a'B', e'B'', which are parallel to Aa. It will often be found convenient to describe this trial polygon below the diagram representing the arch ring, to avoid confusion (*vide Fig.* 19). Let the straight line P$p\pi'$ drawn through the pole P parallel to the closing side, or axis of abscissæ of the polygon, meet Aa in p. The point p is constant for all poles. A point g' in the line of action of the resultant load may be found by producing the directions of the extreme sides or links, 1 and 8 of the polygon, and a second point, by completing the parallelogram Ae, and drawing through π' a straight line parallel to the diagonal Ae (*vide* para. 17).

In the example under consideration, the critical sides are evidently Nos. 1, 4, and 8.

Find the lines of action of the resultants of the forces which are balanced by side 4 (*i.e.*, of the forces A'B'C' on the one side and of D'E'F'G' on the other), in the manner already explained, that is, by drawing through the point of intersection of sides 1 and 4, the straight line $S_3'g_3$ parallel to Aγ, *Fig.* 20, and through the point of intersection of sides 4 and 8, $S_3''g_3$ parallel to γa. Now choose points S_3', S_3'' in these respective lines of action such that, while the straight line $S_3'S_3''$ *almost* touches the upper ring curve, the straight lines drawn through S_3'' and S_3', to *almost* touch the lower limiting curve, may meet on gg_3 (the line of action of the total resultant load), and the consequent pole O yield a polygon fulfilling the requirements already detailed. By "*almost touch*" is meant that the form of the polygon must, in fact, be such that the *curve* corresponding to it, and not the polygon itself, shall touch the limiting curves.

The extreme sides and closing side only of the required polygon are shown in *Plate* IV.

84. An example of each of the simpler cases A and B is given in Section III., Part II.

* It may be sometimes necessary to deal with the resultant of two or more of the forces, for convenience.

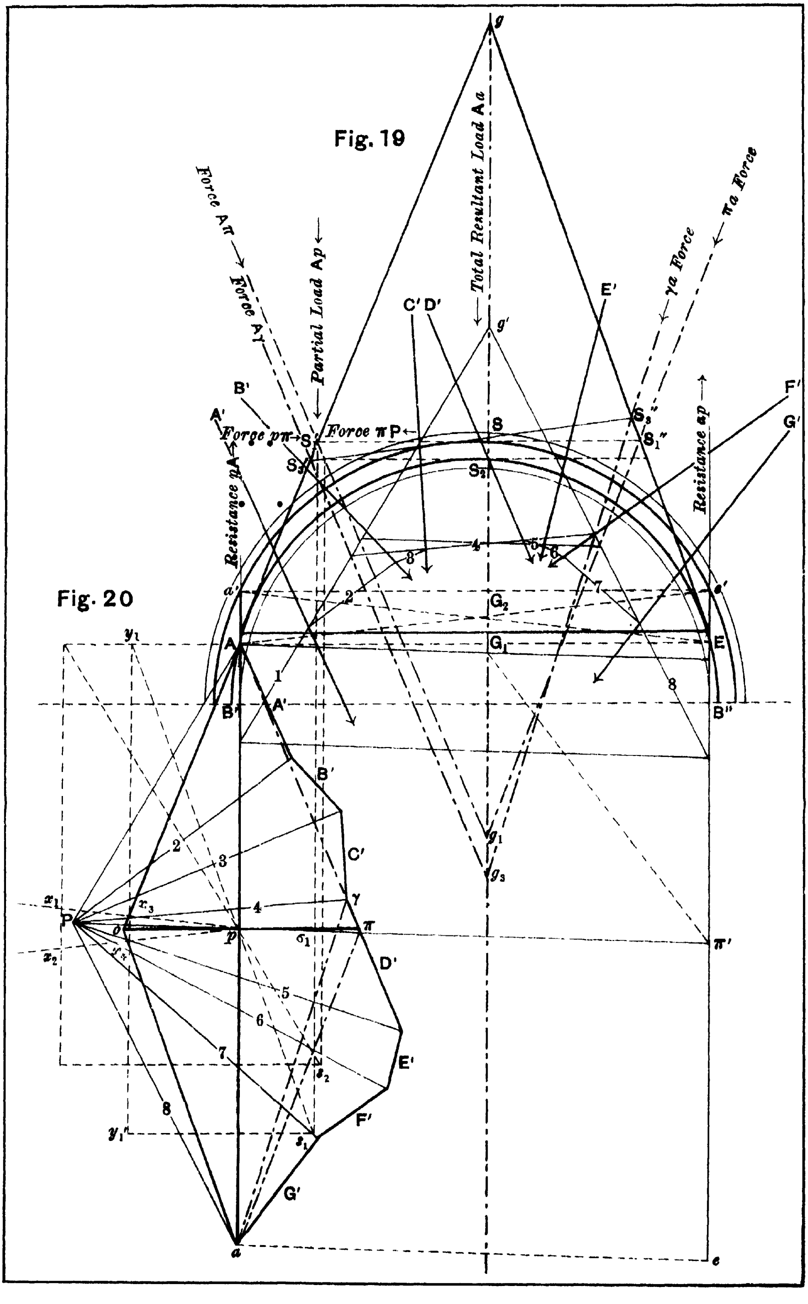


Fig. 19
Fig. 20
Total Resultant Load Aa
Partial Load Ap
Force Aπ
Force Aγ
γa Force
πa Force
Force pπ
Force πP
Resistance pA
Resistance ap

CHAPTER VII.

SPECIAL APPLICATIONS OF THE EQUILIBRIUM POLYGON TO CASES OF PARALLEL LOADING; CENTRE OF GRAVITY AND MOMENTS OF INERTIA.

Re-actions at Points of Support.

85. The following will often be found a convenient method of determining the re-actions at the points of support of a beam, loaded with parallel detached loads :—

Let the beam A′A″, *Fig.* 21, be supposed loaded with detached loads W_1, W_2, W_3, and W_4, in the manner shewn; required to measure the re-actions at the supports.

Set off the loads on any convenient scale and in their proper order from one extremity of the beam, as A′, and at right angles to it, as A′*a*, and, taking the other extremity A″ as pole and also as one extremity of the equilibrium polygon, describe the stress diagram A′*a*A″A′, and the corresponding equilibrium polygon $A''a_4a_3a_2a_1oA''$ in the usual way. It is evident that the segments A′*o* and *oa* represent the re-actions at the points A′ and A″ on the same scale as A′*a* represents the total load.

Centre of Gravity and Centre of Inertia.

86. Before proceeding further, it may be as well to point out that there is a difference between the *Centre of Gravity* of a body and its *Centre of Inertia or Mass*; the two terms not being synonymous.

The *Centre of Inertia or Mass* of a system of material points, whether connected with one another or not, is referred to and defined in Thomson and Tait's *Elements of Natural Philosophy* (1873), para. 195, as "the point whose distance is equal to their average distance from any plane whatever." "The Centre of Inertia or Mass is thus a perfectly definite point in every body or group of bodies. The term *Centre of Gravity* is often very inconveniently used for it. The theory of the

resultant action of gravity shows that, except in a definite class of distributions of matter, there is no fixed point which can properly be called the Centre of Gravity of a rigid body. In ordinary cases of terrestrial gravitation, however, an approximate solution is available, according to which, in common parlance, the term *Centre of Gravity* may be used as equivalent to *Centre of Inertia;* but it must be carefully remembered that the fundamental ideas involved in the two definitions are essentially different."

Determination of the Centre of Gravity.

87. The application of the equilibrium polygon to the determination of the position of the *Centre of Gravity* of a structure is obvious from what has already been explained.

The structure may be supposed divided up into any number of parts by planes conveniently placed, and the mass of each part supposed to be concentrated at the centre of gravity of the latter. The structure may thus be reduced to an equivalent system of material points.

If we suppose the structure divided by parallel planes at unity distance apart, each sectional area may be supposed divided up into triangles, trapezoids, or rectangles, and each of these partial areas reduced to that of a rectangle of constant base-length, the height of which will thus be proportional to the area, and hence to the force of gravity acting on it. Regarding these lengths as forces acting at the centres of figure (*i.e.*, of gravity) of the partial areas, we obtain a system of parallel loads in the line of action of the resultant of which the centre of gravity of the system must lie. This resultant line of action may, as already explained, be determined by means of a force, and an equilibrium, polygon.

Again, if we suppose the same forces to be revolved through an angle, say 90°, while their points of action remain unchanged, we shall obtain a second force, and a second equilibrium, polygon, and so also the line of action of a second resultant, likewise passing through the centre of gravity of the system. The point in which these two lines of resultant action intersect will obviously be the centre of gravity of the system.

If the given area has an axis of symmetry, this may, of course, be regarded as one resultant line of action.

This method, it will be seen, is equivalent to a graphic application of the well known practical rule for finding the centre of gravity of an irregular figure;—"Cut the figure drawn to a large scale out in card board, and suspend it in two positions by a fine thread. The intersection of the two lines of thread corresponds with the position of the centre of gravity of the figure."

If the forces, in the above method, be supposed to be revolved through a right angle, it will be unnecessary to describe a second force polygon; the one first described may be utilized by laying the edge of a set square along its stress lines, and so obtaining a second series of stress lines at right angles to the first set.

88. In *Plate* VI., *Figs.* 24 to 27, an example of this method is given. The points g_1, g_2, g_3, g_4 and g_5, *Fig.* 24, represent the positions of the centres of gravity of five areas, weighted respectively w_1, w_2, w_3, w_4 and w_5. Through the given points axial lines are first drawn at right angles to one another, as shown in *Fig.* 24, and, with any convenient pole P, as in *Fig.* 25, a force polygon is described, and its stress lines numbered consecutively 1, 2, 3, 4, 5 and 6. If the straight lines 1′, 2′, 3′, 4′, 5′ and 6′ be set off at right angles to these, and correspondingly numbered, they will evidently be parallel to the sides of the second equilibrium polygon, which is to be drawn. In any convenient position, now, in prolongation of the lines of action drawn parallel to the direction marked YY, the equilibrium polygon $a_1b_1c_1d_1e_1f_1$ is described, and its resultant line of action YY drawn in. A similar operation is performed for the direction marked XX, and the resultant line of action XX is drawn in by means of the equilibrium polygon $a'b'c'd'e'f'$, *Fig.* 27. In describing these polygons it is only necessary, in order to avoid confusion, to pay attention to the numbering of the bars balancing any particular load. Thus, the bars numbered 1 and 2, as well as those numbered 1′ and 2′, balance the load w_1; similarly bars 4 and 5, as well as 4′ and 5′, balance load w_4; and so on; so that bars 1 and 2, or 1′ and 2′, as the case may be, must meet on the line of action passing through g_1; and bars 4 and 5, or 4′ and 5′, as the case may be, must intersect in the line of action passing through g_4; and so on. The intersection of the resultant lines XX and YY obviously fixes the position G of the centre of gravity of the system.

89. The equilibrium polygons, *Figs.* 26 and 27, evidently enable the positions of the centres of gravity of any group of the given system

of loads to be determined. For instance, if the directions of bars 1 and 3, or 1′ and 3′, be produced, and through their point of intersection straight lines be drawn parallel to XX and YY respectively, then will the intersection g_{1-2} of these straight lines fix the position of the centre of gravity of loads w_1 and w_2. Similarly, axial lines drawn through the intersections of bars 1 and 5, and of 1′ and 5′, will fix the position of the centre of gravity of loads w_1 to w_4; and so on.

The Centre of Gravity of Quadrilateral Figures.

90. It will be well here to introduce the following propositions :—

(1). *To find the centre of gravity of a trapezium.*—Bisect the parallel sides AB and DC, *Fig.* 22, of the trapezium ABCD in the points E and F. Join EF. The required point must lie in the axis of symmetry EF. Join DE, BF, and draw g_1g_2, joining g_1 and g_2 the centres of gravity respectively of the triangles DAB and DCB. The intersection G of the straight lines g_1g_2 and EF is the centre of gravity of the trapezium.

(2). *To find the centre of gravity of an irregular quadrilateral* ABCD, *Fig.* 23.—Draw the diagonals AC, BD; bisect AC in E; join ED, EB, and draw g_1g_2 parallel to BD, through the centre of gravity, either of triangle ADC, or triangle ABC. Take DH = BF, and join HE. The intersection G of HE and g_1g_2 is the required point.

N.B.—Method (1) will generally be found quicker than method (2) for the case of a trapezium.

The graphical equivalent of the expression Σ wx.

91. If the directions of the sides 3, 4 and 5 of the equilibrium polygon of *Fig.* 26 be produced to meet YY in the points γ_1, δ_1 and ϵ_1 respectively, the triangles $\delta_1c_1\gamma_1$, $\epsilon_1d_1\delta_1$, and $f_1e_1\epsilon_1$ so formed will evidently be similar to the triangles 34 W_3, 45 W_4 and 56 W_5 respectively of the corresponding force polygon of *Fig.* 25; and if we put x_3, x_4, x_5 for the distances to the right of YY of the several lines of load w_3, w_4 and w_5, and p for the length of the polar distance P, *Fig.* 25, the following relations will obtain—

$$p : w_3 :: x_3 : \gamma_1\delta_1 \left(= \frac{w_3x_3}{p}\right).$$

$$p : w_4 :: x_4 : \delta_1\epsilon_1 \left(= \frac{w_4x_4}{p}\right).$$

$$p : w_5 :: x_5 : \epsilon_1f_1 \left(= \frac{w_5x_5}{p}\right).$$

Now $\gamma_1\delta_1$, $\delta_1\epsilon_1$, $\epsilon_1 f_1$ are the lengths intercepted on YY by the directions of the sides 4, 5, and 6 of the equilibrium polygon $a_1b_1c_1d_1e_1f_1$, while w_3x_3, w_4x_4, and w_5x_5 measure the moments of the several weights w_3, w_4, and w_5 about the axis YY; and the length of p is arbitrary (say 5, 10, or 100 weight units according to the scale employed); so also if β_1 be the point in which the side a_1b_1 intercepts YY, and we put $-x_2$ and $-x_1$ for the distances *to the left* of YY of the lines of loads w_2 and w_1, we shall have

$$p : w_2 :: -x_2 : \gamma_1\beta_1 \left(= -\frac{w_2\,x_2}{p}\right)$$

$$p : w_1 :: -x_1 : \beta_1 f_1 \left(= -\frac{w_1\,x_1}{p}\right)$$

The above relation is, moreover, evidently quite general; that is to say, that for any axis s_0f_0, taken parallel to the loading, the moments of the loads w_2, w_3, w_4, and w_5, situated to the right of s_0f_0, are proportional to the lengths $y_0\beta_0$, $\beta_0\delta_0$, $\delta_0\epsilon_0$, $\epsilon_0 f_0$ intercepted on s_0f_0 by the corresponding sides of the polygon produced, while the moment of the load w_1, situated to the left of s_0f_0, is proportional to $-y_0a_0$.

Hence, the general theorem,—*The lengths intercepted on any straight line drawn parallel to the direction of the loading by the sides of the equilibrium polygon produced, are proportional to the moments of the corresponding loads measured with regard to that straight line (due attention being paid to the direction in which such moment acts).*

91a. This theorem is especially important in its bearing on the graphical determination of the product and moments of inertia of a system of parallel forces, and it will also be evident that,—granted its truth,—it follows that the equilibrium polygon drawn for any given system of loads applied to a given structure affords a graphic representation of the bending moment at successive sections of the structure whose planes are parallel to the direction of the total resultant load.

For, suppose the polygon, *Fig.* 26, to represent the equilibrium polygon of a given structure, spanning the interval AH, and drawn for *any* given system of loads (para. 10), whose total resultant line of action is f_1f_2. Consider the equilibrium of any section s_0y_0 of the structure, whose plane is (perpendicular to that of the paper and) parallel to f_1f_2. The oblique loads, if any, being resolved at the points in which they meet the frame parallel to f_1f_2 and to the closing side AH, the former components combine with the forces parallel to f_1f_2 to form the resultant

bending couple; those, if any, parallel to AH meet the plane of the section $s_0 f_0$, and combine with the resistances there to form the resultant couple of resistance.

This may be easily proved by reference to paras. 33 and 34, and *Figs.* 1 to 3 of *Plate* I. For the resultant load hc, for instance, which acts on the section $x'y'$ of the structure at the point of intersection of bars 3 and 8 (being the bars which balance it) may be there resolved into the forces kc', *Fig.* 1, parallel to the total resultant load ag, and $hk + c'c$ combining with and acting along stress 8, so that along the closing piece 8 the forces $Ph + hk + c'c$ now act. The resistance of bar 3 is measured by cP, *Fig.* 1, that is, by $c'k$ parallel to ag, and $cc' + kh + hP$ parallel to stress 8; so that the bending couple acting at the section $x'y'$ is the moment of kc', and the resisting couple that of $Ph + hk + c'c$.

Returning now to the theorem under consideration and to *Fig.* 26, the moment of the component, parallel *to* $f_1 f_2$, of the resistance at H —the force, that is, which is balanced by the sides AH and He_1, of the polygon,—is represented by the intercept $s_0 f_0$; the moments of the remaining forces acting to the right of the section $s_0 y_0$ by the intercept $y_0 f_0$; the bending moment, therefore, being measured by the difference of these moments, is represented by the intercept $s_0 y_0$. So also for any other section.

The application of the theorem to cantilevers is self-evident. Any portion of the frame, *Fig.* 26, may be regarded as representing the neutral axis of the equivalent pieces of a cantilever (para. 27). Suppose the portion Ad_1 to be loaded with a given system of loads at the angles a_1, b_1 and c_1 of the frame. The bending moment at any section $s_0 f_0$ of the structure is represented by the intercept $y_0 \delta_0$.

92. It will be observed that the above method affords a graphic method of determining the value of the expression Σwx, where w represents any one of a system of detached loads, and x the distance of its line of action from any axis taken parallel to the latter.

93. It is to be particularly observed that the method described in para. 91 is equally applicable to oblique, as to rectangular, co-ordinates.

The Moments and Products of Inertia of a System of Parallel Loads with regard to a given axis.

94. In para. 91 it is shown that, for a system of parallel loads, the

length intercepted on any given axis (drawn parallel to the direction of the loading) by any two sides of the equilibrium polygon produced, is proportional to the moment of the load, which is balanced by those two sides, with regard to that axis, due attention being paid to the direction in which each moment acts.

If, now, the intercepts so obtained be regarded as representing new values of the corresponding loads, acting at the same points as before and parallel to their former direction, and if a new force polygon be described, with the former, or some more convenient, polar distance, and with the new values of the forces substituted for the former (due regard being paid to their directions), then will the lengths of the new intercepts on the given axis, obtained in exactly the same manner as before (*i.e.*, by describing ah equilibrium polygon, with its angles lying on the directions of the forces, and producing its sides to intercept the given axis) be proportional to the moments of the moments of the original system of parallel loads, taken with regard to that axis; and the sum of such intercepts will, therefore, represent the value of the expression $\frac{\Sigma wx^2}{pq}$, being the Moment of Inertia of the system with regard to the given axis divided by an arbitrary quantity.

95. *Figs.* 28 and 29, *Plate* VI., illustrate this. *Fig.* 28 shows a force polygon, described with polar distance q and forces equal to $f_1\epsilon_1$ $\epsilon_1\delta_1$, $\delta_1\gamma_1$, $\gamma_1\beta_1$, and $\beta_1\alpha_1$ of *Fig.* 26. These forces (para. 91) are the following:—

$f_1\epsilon_1$	represents	$\frac{w_5x_5}{p}$	acting	downwards	at the point	g_5.
$\epsilon_1\delta_1$	"	$\frac{w_4x_4}{p}$	"	"	"	g_4.
$\delta_1\gamma_1$	"	$\frac{w_3x_3}{p}$	"	"	"	g_3.
$-\gamma_1\beta_1$	"	$\frac{w_2x_2}{p}$	"	upwards	"	g_2.
$-\beta_1\alpha_1$	"	$\frac{w_1x_1}{p}$	"	"	"	g_1.

It will be seen that the polygon of external loads $\alpha_1\gamma_1 f$, *Fig.* 28, is a closed one, and that, therefore, the extreme sides of the corresponding equilibrium polygon $a_2b_2c_2d_2e_2$, *Fig.* 29, are parallel. The consequence of this is that the intercepts $f_2\epsilon_2$, $\epsilon_2\delta_2$,...$\beta_2\alpha_2$ of its sides on the axis YY are all *positive*, which agrees with the analytical meaning of the expression Σwx^2. The sum of their lengths, therefore, determines the value of

the Moment of Inertia of the system about the given axis, since the values of p and q are arbitrary, for $\frac{\Sigma wx^2}{pq} = f_2 a_2$.

96. With regard to the lengths of the polar distances it must be carefully borne in mind that, should the intercept be measured in units of length, the polar distance must be reckoned on the scale of loads, and *vice versâ*.

97. If the second force polygon, *Fig.* 28, be now turned until the load line becomes parallel to the XX axis, in exactly the same manner as was employed for determining the centre of gravity of the system, and a new equilibrium polygon, as shown in *Fig.* 30, be described, then will the new intercepts on the axis taken parallel to the line of loads be of the form $\frac{wxy}{pq}$, and their algebraic sum proportional to Σwxy, the Product of Inertia of the system relatively to the axes chosen.

98. The following relation may be noted:—In para. 91, it is shown that the intercept on any axis, parallel to the loading, made by producing two contiguous sides of the equilibrium polygon is equal to the numerical value of the moment, with regard to that axis, of the load balanced by those sides, divided by the polar distance of the corresponding force polygon. For instance, if c_0, *Fig.* 26, be the distance of the line of action of load w_3 from axis f_0y_0, then $\beta_0\delta_0 = \frac{w_3c_0}{p}$. Multiplying both sides of the equation by $\frac{c_0}{2}$ we have $\beta_0\delta_0 \times \frac{c_0}{2} = \frac{w_3c_0^2}{2p}$, or, in words, the area of the triangle $\beta_0c_1\delta_0$ = moment of inertia about f_0y_0 of the load w_3, divided by the arbitrary length $2p$. This relation being true for each one of the loads there follows :—The moment of inertia of the system is proportional to the area of the figure bounded by the given axis (f_0y_0), and by the extreme sides produced of the equilibrium polygon $(a_1a_0f_0e_1a_1)$, less the area of the equilibrium polygon itself $(a_1b_1c_1d_1e_1a_1)$.

99. The moments of inertia of a system of material points about *any* axis is equal to the moment of inertia of the system about a parallel axis passing through its centre of gravity *plus* the moment of inertia of the entire mass, supposed concentrated at its centre of gravity, about that axis.

For if AB, *Fig.* 32, *Plate* V., be any axis about which the moments of inertia of the system are to be measured, and YY a parallel axis passing through G, the centre of gravity of the system, the distance of AB from G being denoted by $\bar{x}$, that of any centre of mass g from AB by x, and from YY by x', then, if m be the mass concentrated at g, we have

$$mx^2 = m(\bar{x} + x')^2 = m(\bar{x}^2 + 2\bar{x}x' + x'^2).$$

And for the system, $\Sigma mx^2 = \Sigma m(\bar{x}^2 + 2\bar{x}x' + x'^2)$.

But, since YY passes through the centre of gravity of the system

$$\Sigma mx' = 0, \text{ therefore } \Sigma mx^2 = \Sigma m\,\bar{x}^2 + \Sigma mx'^2.$$

100. If I_x, I_y denote the moments of inertia of a given system of material points about rectangular axes XX, YY, and I_z, those about an axis ZZ, perpendicular to the plane in which XX, YY lie, and passing through their point of intersection O, *Fig.* 33, then will the relation obtain

$$I_z = I_x + I_y$$

For, consider any centre of mass m, distant r from O, the co-ordinates of which are x and y. The moment of inertia of m about O $= mr^2 = mx^2 + my^2$, which relation, being true for each material point of the system, we have the constant polar relation $\Sigma mr^2 = \Sigma mx^2 + \Sigma my^2$ or $I = I_x + I_y$.

101. If I_x, I_y, denote the moments of inertia, and K_{xy} the product of inertia of a system of material points relatively to rectangular axes YY and XX, passing through the centre of gravity of the system, it is evident that if the axes of co-ordinates be supposed to revolve round their point of intersection, the values of I_x, I_y, and K_{xy} must vary. Those axes for which the value of one of the expressions I_x or I_y is a maximum and the other a minimum are called *Principal Axes*, and for these the value of K_{xy} is zero (para. 105, and Rankine's Applied Mechanics, 3rd Edition, para. 95).

Radius of Gyration.

102. If $I = \Sigma mx^2$ measure the moments of inertia, of a system of material points about an axis, and k be such a length that $k^2\Sigma m = \Sigma mx^2$, then is k called the *radius of gyration* of the system.

Having found the value of Σmx^2 by the method described in Art. 94, it will be seen that that of k may be geometrically determined in the following manner.

It will be remembered that, in order to measure the moment of inertia of a system of parallel forces, we must describe *two* force and equilibrium

polygons, and that if the pole distances be p and q respectively, we have the following relation between the intercept $f_2\, a_2$, *Fig.* 29, cut off from the given axis by the extreme sides of the second equilibrium polygon

$$f_2 a_2 \times p \times q = \Sigma m x^2 = \text{(by hypothesis) } k^2 \Sigma m$$

$$\therefore k^2 = f_2 a_2 \times \frac{p \times q}{\Sigma m}.$$

Now Σm is proportional to, and may be supposed to be represented by, AB, the load line of the first force polygon, *Fig.* 25, and the value of k may therefore be determined by the following construction*:—

Set off Ar, *Fig.* 25, equal to the second polar distance q, and draw $r\pi$ parallel to BP, and πt parallel to the direction in which the second pole distance p is measured, that is, perpendicular to AB.

Then $$\frac{\pi t}{p} = \frac{\mathrm{A}r}{\mathrm{AB}} = \frac{q}{\Sigma m}$$

$$\therefore \pi t = \frac{p \times q}{\Sigma m}$$

and, therefore, $$k = \sqrt{f_2 a_2 \times \pi t}.$$

The construction of *Fig.* 31 for finding a mean proportion between the lengths $f_2 a_2$ and πt is obvious.

103. In a similar way a length c may be found such that $c^2\, \Sigma m = \Sigma mxy$, or an area C, equal to the square on c, may be found, equivalent to the linear portion of the expression for the product of inertia.

104. *Given the radii of gyration of a system of material points about rectangular axes passing through a given origin, required the radii of gyration of the same system about rectangular axes having the same origin but making an angle θ with the former axes.*

Let C, *Fig.* 34, be the origin, CX and CY the old, and CA and CB the new, axes of co-ordinates. Let a^2, b^2, c^2, be the radii of gyration referred to the old, a'^2 b'^2 and c'^2 those referred to the new, position of the axes.

If x, y be the old co-ordinates of any point, of mass m, the new co-ordinates x' y' are related to them as follows :—

It is evident from *Fig.* 34 that

$$y = x \tan \theta + \frac{y'}{\cos \theta}$$

$$\therefore y' = y \cos \theta - x \sin \theta, \quad \text{......................} \quad (1).$$

Also, $x' = \dfrac{x}{\cos \theta} + y' \tan \theta$; therefore, substituting for y' we have

$$x' = x \cos \theta + y \sin \theta, \quad \text{.........................} \quad (2).$$

* Dubois' "Graphical Statics," 3rd Edition, p. 64.

Squaring (1) and multiplying by m, we have

$$my'^2 = my^2 \cos^2\theta - 2 \cos\theta \sin\theta \, mxy + mx^2 \sin^2\theta.$$

$$\therefore \Sigma my'^2 = \cos^2\theta \, \Sigma my^2 - 2\cos\theta \sin\theta \, \Sigma mxy + \sin^2\theta \, \Sigma mx^2$$

$$\text{or } a'^2 \Sigma m = a^2 \cos^2\theta \, \Sigma m - 2c^2 \cos\theta \sin\theta \, \Sigma m + b^2 \sin^2\theta \, \Sigma m$$

$$\therefore a'^2 = a^2 \cos^2\theta - 2c^2 \cos\theta \sin\theta + b^2 \sin^2\theta$$

$$= \{a^2 - 2c^2 \tan\theta + b^2 \tan^2\theta\} \cos^2\theta.$$

$$\therefore a' = \left\{\left(b \tan\theta \sim \frac{c^2}{b}\right)^2 + \left(a^2 - \frac{c^4}{b^2}\right)\right\}^{\frac{1}{2}} \cos\theta, \ldots\ldots\ldots\ldots (3).$$

$$\text{Similarly } b' = \left\{\left(a \tan\theta + \frac{c^2}{a}\right)^2 + \left(b^2 - \frac{c^4}{a^2}\right)\right\}^{\frac{1}{2}} \cos\theta, \ldots\ldots\ldots\ldots (4).$$

Now the expression under the radical sign may be easily constructed geometrically, as follows:—

Set off the given radii of gyration a and b above and below, and to right and left of, the origin C, along the X and Y axes, and through their extremities draw straight lines parallel to the axes, thus forming the rectangle XYX′Y′, *Fig.* 34. Describe a semicircle on two contiguous sides of the rectangle, so that the radius of one is a, and that of the other b (as shown in *Fig.* 34).

If CK be the new direction of the X-axis, making an angle θ with CX, we have $\text{KX} = a \tan\theta$, and $\text{Y}'\text{K}' = b \tan\theta$. If, then, the known quantities $\frac{c^2}{a}$ and $\frac{c^2}{b}$ be set off, the former downwards from X to G, the latter to right of Y′ from Y′ to G′, we have $\text{GK} = a \tan\theta + \frac{c^2}{a}$ and $\text{G}'\text{K}' = b \tan\theta - \frac{c^2}{b}$. But $\text{XH} = b$, and $\text{Y}'\text{H}' = a$, so that $\text{GH} = \sqrt{b^2 - \frac{c^4}{a^2}}$ and $\text{G}'\text{H}' = \sqrt{a^2 - \frac{c^4}{b^2}}$. Therefore $\text{KH} = \left\{\left(a \tan\theta + \frac{c^2}{a}\right)^2 + \left(b^2 - \frac{c^4}{a^2}\right)\right\} = b' \sec\theta$, and $\text{K}'\text{H}' = \left\{\left(b \tan\theta \sim \frac{c^2}{b}\right)^2 + \left(a^2 - \frac{c^4}{b^2}\right)\right\}^{\frac{1}{2}} = a' \sec\theta$.

The construction of *Fig.* 35 for finding the quantities $\frac{c^2}{a}$ and $\frac{c^2}{b}$ is obvious. Set off $\text{CD} = c$ at right angles to $\text{AD} = a$, and draw CB at right angles to AC to meet AD produced in B. Then $\text{DB} = \frac{c^2}{a}$, and since

$$\frac{c^2}{a} : \frac{c^2}{b} :: b : a$$

if DC be produced to E, so that $\text{DE} = \text{AD} = a$, and AB be produced to F, so that $\text{DF} = b$, and FE be joined, and BG drawn parallel to FE to meet ED in G, then

$$\mathrm{DF} : \mathrm{DE} :: \mathrm{DB} : \mathrm{DG}$$
$$\text{or } b : a :: \frac{c^2}{a} : \frac{c^2}{b}.$$

Hence, if semicircles be described with centres K and K′, *Fig.* 34, and respective radii KH and K′H′ to cut the axes drawn through X and Y in the points k, k, and k', k', and through the pairs of points straight lines be drawn parallel to the new axes, *i.e.*, through the points k, k straight lines parallel to CK and through the pair k', k', straight lines parallel to CK′, then will the new radii of gyration CA or CA′ and CB or CB′ be determined.

In this way, by taking a series of points, as Q, joining Q to H, and describing the arc to meet the axis drawn through X in the points k_1 k_1, the radius of gyration Cq' may be determined; and likewise by producing Cq' to meet the axis drawn through Y′ in Q′ joining Q′ and H′, and proceeding as before, the radius Cq may be determined.

In this way the locus of the extremities of the radii of gyration of the given system for different positions of the axes CX and CY might be determined. It is known as the *Curve of Inertia*, and, for a system of material points, will be found to be a closed curve. It may be shown to be an ellipse as follows :—

105. *To determine the locus of the extremities of the radii of gyration, for a given system of material points, drawn from any given point as centre.*—By supposing three planes of reference to pass through the given point, and the several centres of mass of the given system to be referred to them, we may, by dealing with that one of them containing the X-Y axes, find the radii of gyration a, b, and c, of the given system with regard to them, and form equations (3) and (4) of the previous paragraph. As a similar operation may be performed for the X-Z axes with similar results, it will be sufficient to confine our attention to the former plane. Equations (3) and (4) of the previous paragraph may be written—

$$a^2 \cos^2 \theta - 2c^2 \cos \theta \sin \theta + b^2 \sin^2 \theta - a'^2 = 0, \ldots\ldots\ldots (5).$$
$$a^2 \sin^2 \theta + 2c^2 \cos \theta \sin \theta + b^2 \cos^2 \theta - b'^2 = 0, \ldots\ldots\ldots (6).$$

Now the curve is, by hypothesis, a *central* one, and the equations remain unaltered by writing $-a'$, and $-b'$ for $+a'$ and $+b'$, that is, the points answering to equal and opposite values of a', b' are equidistant

from the given central point, or origin; a and b are the semi-lengths of chords which correspond with the given axes of co-ordinates X and Y, and may, therefore, together with c be regarded as constants, and the same quantities, in both equations : a' and b' are, by hypothesis, lengths of semi-chords measured at right angles to axes inclined respectively at $90° + \theta$ and θ to the X-axis, a' being inclined at θ and b' at $90° + \theta$ to that axis.

If then we write k for the variable semi-length of a chord, or radius of gyration, and $90° + \theta$ for θ in equation (6), each equation reduces to the form

$$a^2 \cos^2 \theta - 2c^2 \cos \theta \sin \theta + b^2 \sin^2 \theta - k^2 = 0, \quad \text{.........} (7),$$

which may be regarded as the polar equation to the required locus.

Now the general equation of a central curve of the second degree, referred to the origin as centre, is*

$$ax^2 + 2\,hxy + by^2 + c'' = 0, \quad \text{...................} (8),$$

which may be reduced to the polar form by writing $x = \rho \cos \theta$ and $y = \rho \sin \theta$, ρ being any radius vector drawn from the centre as pole. The form of the general equation then becomes

$$a^2 \cos^2 \theta - 2h \cos \theta \sin \theta + b \sin \theta + \frac{c'}{\rho^2} = 0, \quad \text{..........} (9),$$

which, being compared with equation (7), would lead to the conclusion that the required locus is either an ellipse or an hyperbola, the length of any radius vector of which is inversely proportional to the radius of gyration of the system measured with regard to that radius vector as axis.

If, now, the general equation (8) be transformed to one expressing the equation to the curve referred to a pair of conjugate diameters as axes of co-ordinates, the co-efficient h of the term involving xy vanishes,† and, the co-ordinates being by hypothesis rectangular, these conjugate axes are, in fact, *the* axes of the curve, and the Principal Axes of the System.

Hence, *for the Principal Axes of the System, that is, those about which the Moment of Inertia has either a maximum or minimum value, the Product of Inertia vanishes.*

* Todhunter's Conic Sections, 5th Edition, para. 270, or Salmon's Conic Sections, 5th Edition, para. 152.

† Salmon's Conic Sections, 5th Edition, para. 143.

Equation (7) then takes the form

$$a^2 \cos^2 \theta + b^2 \sin^2 \theta - k^2 = 0,$$

or, writing $x = k \cos \theta$ and $y = k \sin \theta$, and A and B for the maximum or minimum values of a and b,

$$Ax^2 + By^2 - k^4 = 0, \qquad (10),$$

which, since A and B are essentially positive, seeing that the forces all act in the *same direction*, represents the equation to an ellipse.*

Hence, the general statement made in para. 237 of Thomson and Tait's "Elements of Natural Philosophy," (1873 Edition).

"For every rigid body there may be described about any point as centre, an ellipsoid (called Poinsot's Momental Ellipsoid) which is such that the length of any radius vector is inversely proportional to the radius of gyration of the body about that radius vector as axis. The axes of the ellipsoid are the *Principal Axes* of inertia of the body at the point in question.

"When the moments of inertia about two of these are equal, the ellipsoid becomes a spheroid, and the radius of gyration is the same for every axis in the plane of its equator. When all three principal moments are equal the ellipsoid becomes a sphere, and every axis has the same radius of gyration."

106. Now, the angle ϕ, through which the rectangular axes of equation (8) must be turned, in order that it may express the equation of the curve referred to its principal axes, is such that

$$\tan 2\phi = \frac{2h}{a-b} \dagger$$

Hence, in order to measure the radii of gyration of the system with regard to the Principal Axes by the method of para. 104 we must suppose the X-axis turned through an angle ϕ, such that

$$\tan 2\phi = \frac{2c^2}{a^2 - b^2} = \frac{2c^2\Sigma m}{a^2\Sigma m - b^2\Sigma m} = \frac{2\,\Sigma mxy}{\Sigma mx^2 - \Sigma my^2}$$

the denominator and numerator of which expression are known by paras. 94 and 97.

107. The contents of this Chapter should be compared with paras. 207 to 209 of Volume I.

* *Vide* Dubois' "Graphical Statics," 3rd Edition, para. 56.

† Salmon's Conic Sections, 5th Edition, para. 156, or Todhunter's Conic Sections, 5th Edition, para. 271.

Fig. 21.

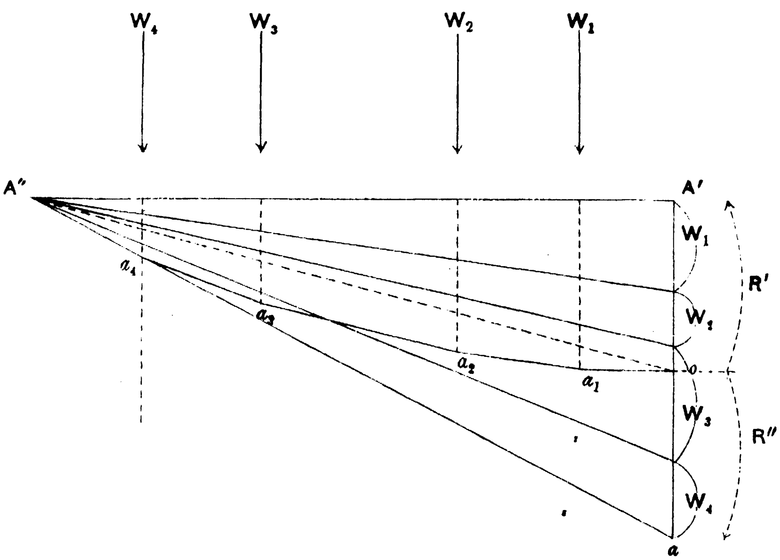


Fig. 23.

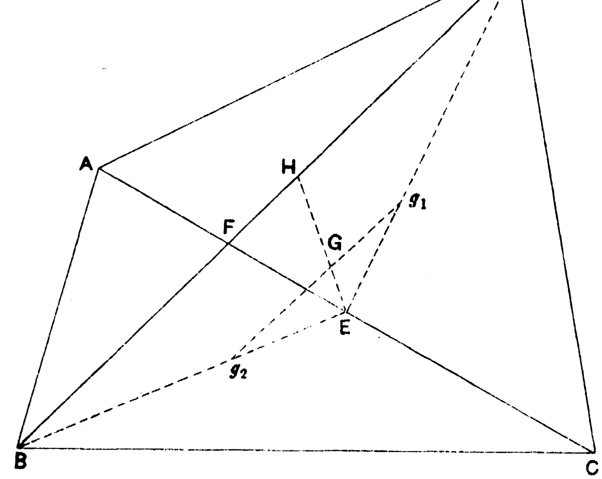


Fig. 22.

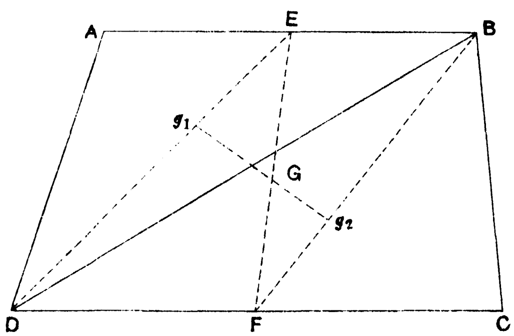


Fig. 32.

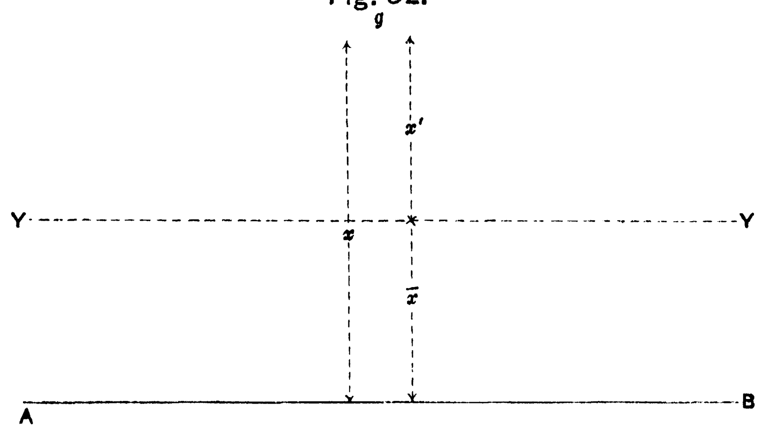
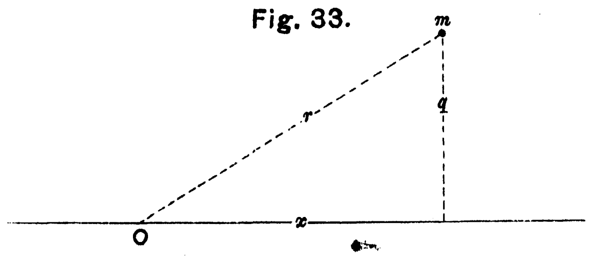


Fig. 33.



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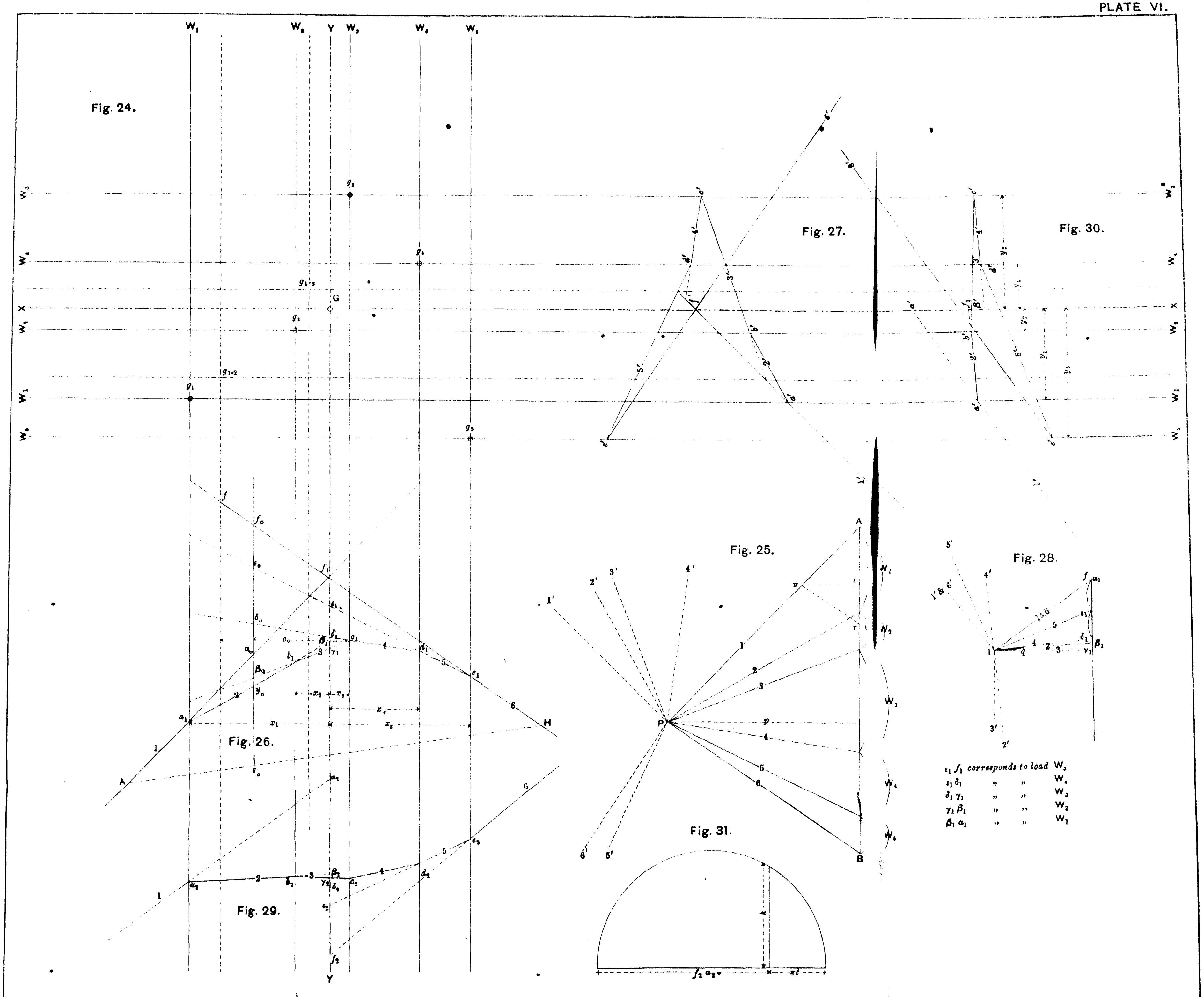

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Fig. 34.

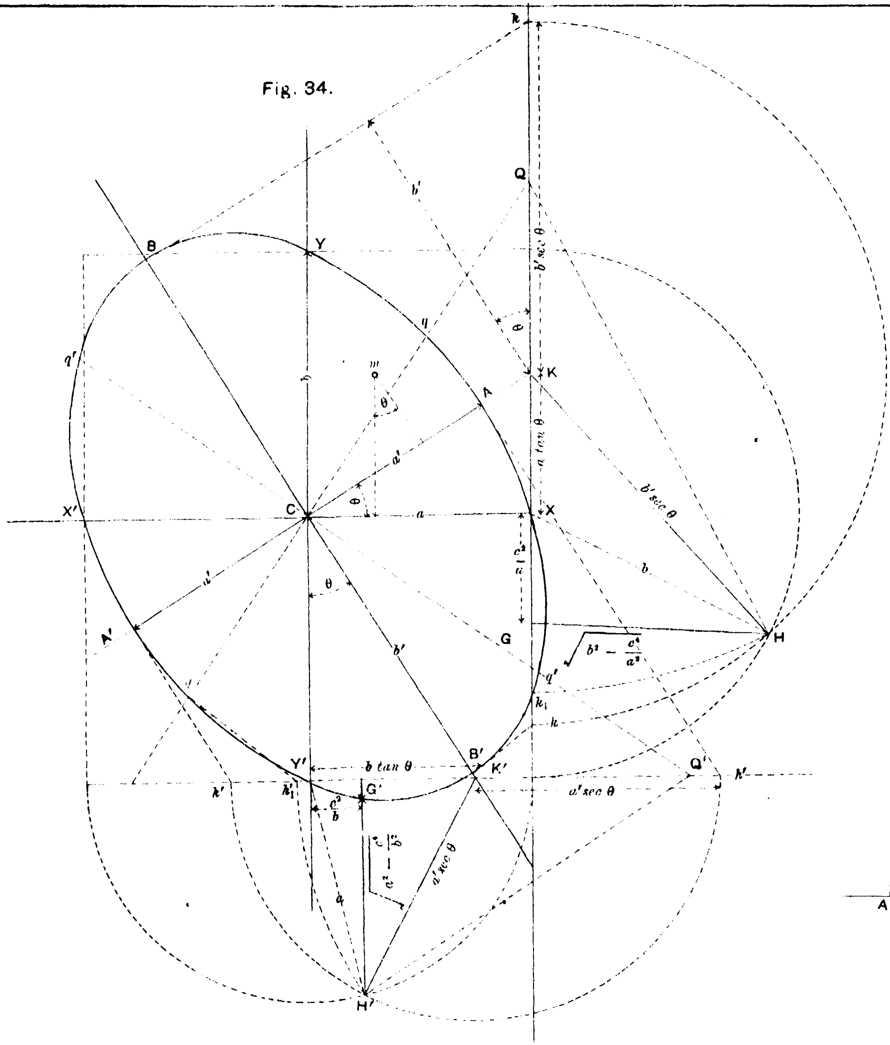
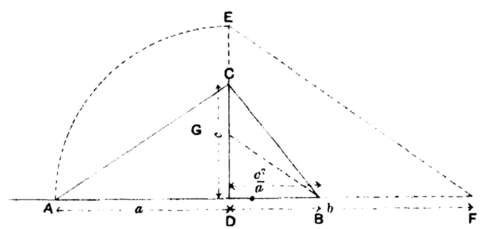


Fig. 35.



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ADDENDUM TO CHAPTER VII.

EXAMPLES IN MOMENTS OF INERTIA.

Moments of Inertia of bodies generally are expressed in pounds and feet, but for the determination of Moments of Resistance to Flexure, Deflection Curves, &c., the hypothetical Moments of Inertia of areas about straight lines in their planes only are required ; for this purpose, supposing the area to be a plane and to have no thickness, the square of the unit of length would be regarded as the hypothetical unit of load. The following two examples will serve to illustrate what is meant :—

Example I. *Fig.* 35*a*, *Plate* VIIA, shows the cross section of the Cross girders of the Plate Iron Railway Bridge designed in Example II. of Chapter XXVII., Section III.

A scale of $\frac{1}{50}$ inch to 1 load unit is taken as the Load Scale, and one of $\frac{1}{10}$ inch to 1 inch as the Lineal Scale. *Fig.* 35*b* shows the moment polygon described in the usual way for a pole distance p; the position of the straight line YY through the centre of gravity of the figure is thus determined, and the lengths of the intercepts on it, giving the values of the moments of the several loads about YY, obtained. Thus, the moment of w_1 about YY is measured by $\alpha_1\,\beta_1 \times p$; that of w_2 by $\beta_1\,\gamma_1 \times p$; and that of w_3 by $\alpha_1\,\gamma_1 \times p$. Now each of these moments is made up of a certain number of load units multiplied by a certain number of lineal units, so that if one component of each (say $\alpha_1\,\beta_1$, $\beta_1\,\gamma_1$ or $\alpha_1\,\gamma_1$) be measured on the scale of loads, the other component (p) must be measured on that of length, and *vice versâ*, it being immaterial which component is measured on which scale, as explained in para. 96.

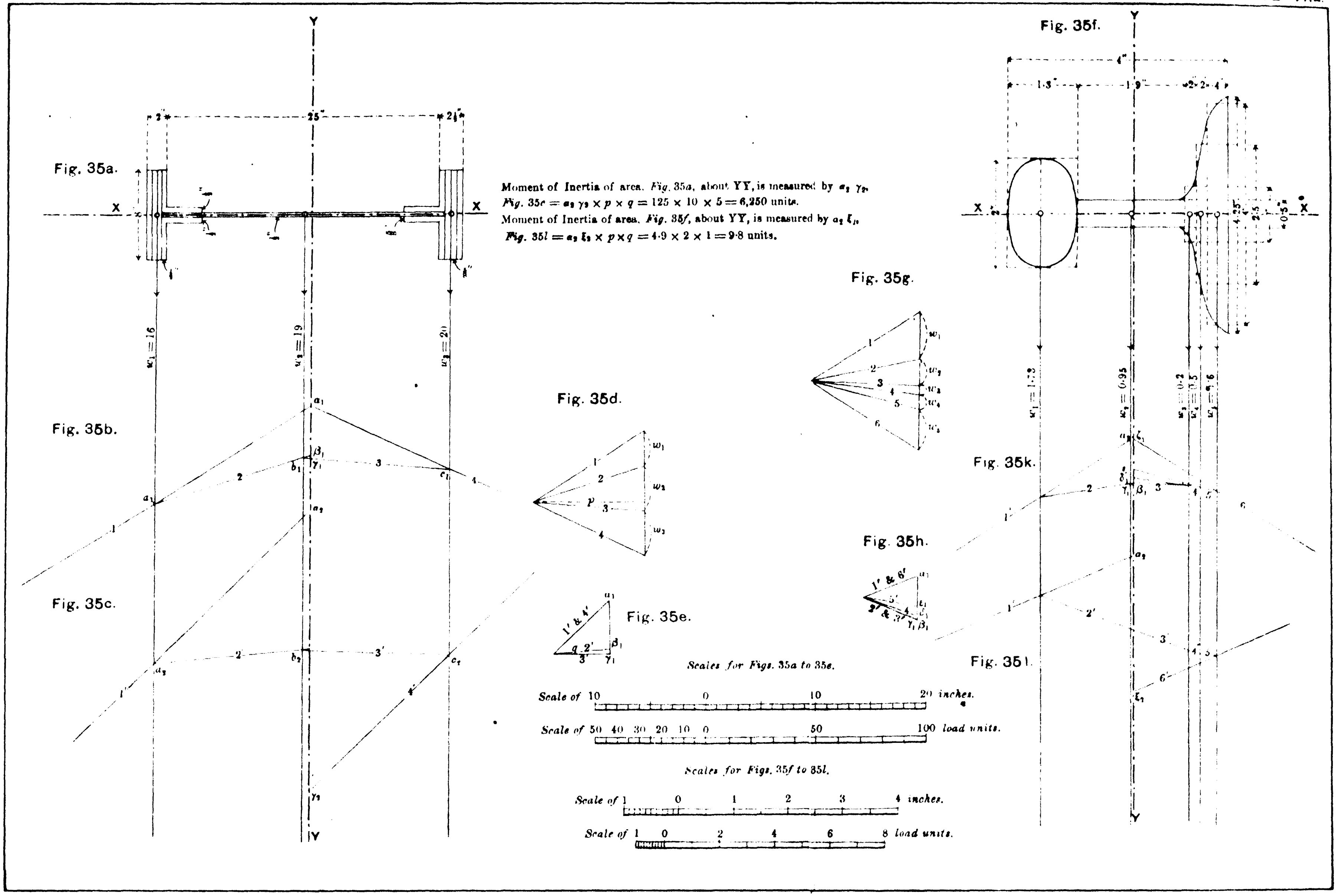
In order to determine I, (the Moment of Inertia of the area,) the lengths $\alpha_1\,\beta_1$, $\beta_1\,\gamma_1$, and $\gamma_1\alpha_1$ are taken to form the load line of a second force polygon, *Fig.* 35*e*, with any convenient pole distance q, and the corresponding equilibrium polygon, which will be an open one, described as in *Fig.* 35*c*. The numerical value of I, as already explained, will then be equal to $\alpha_2\,\gamma_2 \times p \times q$, and the only question is, on what scale are

these three components to be measured. Now the quantity I is of the form $m\Sigma x^2$, being made up of a certain number of load units, multiplied by length units, multiplied by length units. One component of the expression for I must, then, be measured on the scale of loads, and the other two on that of length.

Now $a_2\ \gamma_2$ measures 125 load units about, p measures 10 length units, and q, 5 length units.

Therefore, $I = 125 \times 10 \times 5 = 6{,}250$ units of Moments of Inertia.

Example II. is Example III. of para. 208, page 224, Vol. I., and needs no further explanation.



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CHAPTER VIII.

THE EQUILIBRIUM POLYGON IN ITS APPLICATION TO THE DEFLECTION CURVE.

108. In para. 52 it has been explained that the curve of bending moments of a beam "exhibits the form which a perfectly flexible, weightless and inextensible, string would assume, whose length is equal to that of the curve," were it subjected to the conditions of loading imposed on the beam itself. We propose in this Chapter to investigate as briefly as possible, and by geometrical methods, the form which the partially stiff beam or girder itself will assume, depending, as it does upon the elastic properties of the material of which the beam or girder is composed, as fully explained in Chapter XV., Vol. I., the contents of which should be compared with those of this Chapter.

The following geometrical interpretation (due to Professor Mohr) of the equation to the elastic curve, given in para. 284 of Vol. I., will be examined and applied to cases of beams—supported, continuous and fixed—and cantilevers.

109. The equation, which connects the curvature assumed by the neutral axis of a slightly bent beam at any section under a system of vertical loads with the bending moment M produced at that section, is shown in para. 284 of Vol. I., to be of the form

$$M = \frac{E \times I}{\rho}$$

where ρ is the radius of curvature of the neutral axis of the beam, I the moment of inertia of the cross section in question, and E the modulus of elasticity of the material of which the beam, or girder, is composed.

From this equation we deduce the conclusion that according as M is positive, negative, or zero, ρ is positive, negative, or infinite, and the elastic line either curved downwards, curved upwards, or straight.

If one extremity, the left say, of the beam be taken as origin of co-ordinates, and horizontal distances be denoted by x, and vertical by

y, we have (*vide* para. 285 of Vol. I.) for the differential equation to the elastic line

$$\frac{d^2y}{dx^2} = \pm \frac{M}{EI}, \quad \text{..........................(1).}$$

Consider first the case of beams of uniform cross section. I being invariable for all sections, and E always constant for the particular material of the beam, we have by integration

$$\frac{dy}{dx} = \pm \frac{1}{EI} \int M dx \pm \text{constant}, \quad \text{....................(2).}$$

Now $\frac{dy}{dx}$ measures the ratio which the increment of ordinate bears to that of abscissa of the curve, that is, it measures, at any point, the tangent of the angle which the tangent to the elastic curve at that point makes with the x-axis; while the expression $\int M dx$ denotes the area of the curve whose ordinate at any point represents M, (the bending moment)

If, then, an equilibrium curve or polygon be described in the usual way for the given loading, by means of a force polygon with polar distance p, the following relation, it will be remembered, obtains between its ordinate y at any point x, y, and the bending moment M at that point (Chap. II.), *viz.* —

$$p \times y = M,$$

so that $\int M dx$ becomes $p \int y dx$, and the differential equation to the elastic curve becomes

$$\frac{dy}{dx} = \pm \frac{p}{EI} \cdot \int y dx \pm \text{constant}, \quad \text{.................. (3).}$$

Now $\int y dx$ represents *the area of the curve of bending moments between any given limits*, and in order to approximate to it we may suppose the curve reduced to a polygon, and the polygon divided up by straight lines drawn parallel to y, and the simple areas thus obtained again reduced to rectangles, all of the same base length a, so that their heights become proportional to the simple areas themselves; and we may then suppose these heights to be substituted for the simple areas acting at the respective centres of gravity of the latter, and, in this manner, practically reduce the area of the equilibrium curve, or polygon, to an equivalent system of parallel forces, each acting at a definite point.

The expression $\int y dx$ will thus take the form $a\Sigma h$, in which Σh re-

presents the sum of the heights of the reduced rectangles between given limits. Instead of a curve, we shall now deal with a deflection polygon, the differential equation to which at any vertical boundary of a partial area is of the form

$$\frac{\Delta y}{\Delta x} = \pm \frac{p \times a}{\mathrm{E} \times \mathrm{I}} \Sigma h \pm \text{constant}, \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Putting $\tan a$ for this constant, (which quantity is again referred to in the Addendum to this Chapter), equation (4) becomes

$$\frac{\Delta y}{\Delta x} = \pm \tan a \pm \frac{p \times a}{\mathrm{E} \times \mathrm{I}} \Sigma h, \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and equation (2) becomes

$$\frac{dy}{dx} = \pm \tan a \pm \frac{1}{\mathrm{E} \times \mathrm{I}} \int \mathrm{M} dx, \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

If, then, a second force polygon be described with Σh as load line and $\frac{\mathrm{E} \times \mathrm{I}}{p \times a}$ as pole distance, the tangent of the angle of inclination of any one of its vectors to the direction of the x-axis will be of the form $\Sigma h \div \frac{\mathrm{E} \times \mathrm{I}}{p \times a}$ (as is more fully explained in para. 117), and if the corresponding equilibrium polygon be described, its sides will obviously be parallel to these vectors.

Let this second equilibrium polygon be described, and its corresponding curve be interpolated in the usual way between the points of contact of the tangents. The equation of the curve will be obtained by integrating equation (6) and will be as follows:—

$$y = \pm x \tan a \pm \frac{1}{\mathrm{E} \times \mathrm{I}} \int \left(\int \mathrm{M} dx \right) dx \pm \text{constant}, \ldots\ldots(7)$$

Now, if $x = 0$, $y = \pm$ some constant length, *viz.*, the distance of the origin from some fixed straight line drawn parallel to x-axis above or below it. If the origin be, as supposed, at the left point of support, this constant $= 0$, and equation (7) then becomes

$$y = \pm x \tan a \pm \frac{1}{\mathrm{E} \times \mathrm{I}} \int \left(\int \mathrm{M} dx \right) dx, \ldots\ldots\ldots\ldots(8).$$

This is the equation to the Deflection Curve itself.

When the cross section of the beam is variable, the value of I varies with the cross section, and the form of the equation must be modified by putting I within the sign of summation, thus—

$$y = \pm x \tan a \pm \frac{1}{\mathrm{E}} \int \left(\int \frac{\mathrm{M}}{\mathrm{I}} dx \right) dx.$$

For this case Students are referred to para. 168 *et. seq.* of Chalmers' "Graphical Determination of Forces in Engineering Structures."

110. The principles, then, on which the Elastic, or Deflection Curve of a beam or girder may be described for any system of loads, and for any position of the points of support, provided the cross section of the beam is invariable, are briefly as follows :—

First draw a curve or polygon of bending moments in the usual way; then divide the area of this curve or polygon up by verticals into a number of smaller and simpler areas; and then reduce these simple areas to rectangles, all of the same base length, and regard the heights of the rectangles, so obtained, as forces acting at the centres of gravity of the simple areas. By means of these hypothetical forces we are enabled to determine, in the ordinary way, an equilibrium polygon, which is tangential to the required elastic curve at the points in which the verticals which separate the areas, meet it, and it will be seen that we can thus approximate to the form of the elastic curve as closely as we please by making the verticals sufficiently numerous.

111. The application of the above method to the determination of the deflection curve of a uniform beam, freely supported at its extremities and loaded with a detached load at an intermediate point, will render this explanation more easy of comprehension.

Fig. 36 shows Ex. 4, para. 182 of Vol. I., treated by this method. A beam of length l between points of support at A′, A″, is loaded with a weight W at a point Q distant x_1' from A′ and x_1'' from A″.

A force polygon described in the usual way, with any convenient pole distance, will enable the equilibrium polygon of bending moments A″qA′ to be drawn. The value of the maximum moment at Q is figured y in *Fig.* 36.

The area of this moment polygon A″qA′ equals $\frac{l}{2} \times y$, so that if half the distance between the points of support, or $\frac{l}{2}$, be taken as the base of reduction, we have the total area proportional to length y, which may therefore be taken as the length of the load line in the second force polygon, as shown in *Fig.* 37.

The area of the polygon of bending moments, *Fig.* 36, may be conveniently divided into the two triangles A′qQ and QqA″, but if greater

accuracy in the resulting deflection curve be required, the divisions of the area by vertical straight lines may be made more numerous. The right triangular area is marked I. and the left II. If the load line of the second force polygon, therefore, be laid off to one side of the polygon of bending moments, as shown by the straight line $ab = y$ of *Fig.* 37, it may be divided at once into segments bc and ca, which will be proportional to the triangular areas II. and I. respectively, that is, to the lengths x_1'' and x_1' (since the triangles have the same base length Qq) by the method indicated by dotted lines.

If a pole distance $= \mathrm{E} \times \mathrm{I} \div \frac{l}{2}$ be taken for the load line ab, and a force polygon and corresponding equilibrium polygon, as in *Figs.* 37 and 38, be described in the usual way, the latter will be the tangential envelope of the deflection curve, or elastic line, of the beam. In *Fig.* 37, a trial pole o' was first taken, and the corresponding equilibrium polygon a'' 3 2 1 A′ of *Fig.* 38 drawn, and in this way the point d in the load line ab determined, dividing it into the segments bd and da, which are proportional to the hypothetical resistances at the points of support A″ and A′, respectively. A straight line drawn through d parallel to A′ A″ determines the pole o of the force polygon which yields the required equilibrium polygon A″ 3 2 1 A′. The trial polygon is shown by dotted, the final one by firm, lines in *Fig.* 38. The *curve* of deflection can now be interpolated in the usual way.

112. *Fig.* 40 exhibits the application of the same method to the determination of the deflection curve of a cantilever A′A″, fixed at A′ and loaded with a weight W at A″, being Ex. 1, para. 182, of Vol. I. The lettering is similar to that employed in the previous example. The length y is taken to represent the maximum bending moment acting at A′, and the area of the moment-triangle A″a'A′ is figured II. The force polygon of the second equilibrium polygon is shown to the right of A′, and its load line is made equal in length to y.

113. The case of a supported beam, or of a cantilever, subjected to uniform load, Ex. 2 and 8 of para. 182, Vol. I., would be treated in a manner exactly similar to that already explained, the only difference being that in each of these cases the curve of bending moments being the portion of a parabola, its area would be measured by $\frac{2}{3}$ the corresponding rectangle, instead of $\frac{1}{2}$, as in the case of a triangle.

Remarks.

The following remarks in regard to the above explanation demand special attention:—

114. *Directions of Hypothetical Forces.*—In applying the method of para. 109 to the case of beams freely supported at their extremities, since the value of the bending moment is zero at the end sections and gradually increases to a maximum at some intermediate one, the hypothetical forces of the second series of polygons obviously all act in the same direction. Such, likewise, is the case with cantilevers. But it will be seen in the next Chapter that the forces, corresponding to moment areas in the case of continuous and fixed beams, do not all act in the same direction, although parallel.

115. *Line of action of Hypothetical Resultant Load.*—It will be remembered that the extreme sides of equilibrium polygons, if produced, meet on the line of action of the total resultant load. If, then, sides 1 and 3 of the polygons, shown in *Fig.* 38, be produced, the resultant load I. + II. will pass through their point of intersection. The distance of each component load, therefore, from this resultant load will be in the inverse ratio of their respective magnitudes, or—

Distance of load I. from line of action of I. + II. : distance of load II. from same line : · load II. : load I. : : x_1'' : x_1', but, total distance between loads I. and II. is $\frac{1}{3}$ $(x_1' + x_1'')$.

Hence, distance of load I. from line of action of resultant load I. + II. $= \frac{1}{3}\, x_1''$.

That of load II. from the same line $= \frac{1}{3}\, x_1'$, as figured in *Fig.* 38.

116. *Intercepts on Verticals.*—If the sides of the polygon of bending moments of *Fig.* 36 be produced to meet the verticals through the points of support A″ and A′ in the points *a″* and *a′* respectively, then will the intercept A″ *a″* be proportional to the moment of the weight W about the vertical through A″, and the intercept A′*a′* to that about the vertical through A′ (para. 91). It may be noted that A″*a″* also measures the moment of the resistance at A′ about the vertical through A″ on the same scale as it represents the moment of the load W about the same vertical. These moments are, therefore, equal, and being opposite in direction, balance, as we know to be the case. A similar relation may be established regarding the moments about the vertical through the

extremity A′, and also regarding the intercepts on a vertical drawn through any point made by producing the sides of the second, or deflection, equilibrium polygon.

117. *Equation* (6) *of para.* 109.—That equation (6) does really express the tangent of the angle which the tangent to the deflection curve makes with the x-axis at any chosen vertical section of the beam will be evident if the inclinations of the corresponding vectors of the force polygon be examined.

Fig. 39 shows an enlargement of the force polygon, whose pole is o', of *Fig.* 37. If $o'e$ be drawn perpendicular to the load line, then will $o'e$ represent the pole distance $= \frac{\text{E} \times \text{I}}{p \times a}$.

Consider the inclination of stress 2. The tangent of the angle of inclination of stress 2 to the x-axis, *Fig* 39, $= \frac{ce}{eo'} = \frac{cb - be}{eo'} = \frac{cb}{eo'} - \frac{be}{eo'}$. Now cb represents the height of the reduced triangle II., and, therefore, answers to the symbol Σh of equation (5), or $\int \text{M}dx$ of equation (6), while $\frac{be}{eo'}$ is the tangent of the angle of inclination to the x-axis of side 3, that is of the tangent to the deflection curve at the origin A″.

118. *Pole distance for Deflection Polygon.*—In the above investigation the pole distance for the second force polygon has been supposed to be taken $= \frac{\text{E} \times \text{I}}{p \times a}$. This is necessarily a very large quantity, and the ordinate of the corresponding elastic curve, or the *deflection*, consequently a very small one. But if k be the pole distance of the force polygon and y the ordinate of the equilibrium polygon corresponding to the value M of the bending moment, then is $k \times y = \text{M}$, and, therefore, also $\frac{k}{n} \times ny = \text{M}$, where n is a positive integer, so that any convenient fraction of k may be employed for pole distance provided the resulting ordinate of the equilibrium polygon be correspondingly increased. The pole distance of the second force polygon is, for convenience, taken equal to one-third the span of a supported beam, and one-third the chosen span of a continuous beam, as will be fully explained in the next Chapter, a reduction being afterwards applied to the lengths of the ordinates in accordance with the following paragraph.

A pole distance $= \frac{l}{3}$ has been taken for *Figs.* 37 and 40.

The ratio of the ordinates of the true to those of the constructed Deflection Curve.

119. In paras. 111 and 118, it is stated that *any* convenient length p may be taken as pole distance of the first force polygon, and one-third the span (or selected span) for that of the second force polygon. Since the latter length should be EI, it is evident that the ordinates of the constructed curve will very much exceed those of the true curve.

Since the loads are all parallel, the differences in length of ordinates may be regarded as a simple case of parallel projection (para. 51), the scale of abscissæ remaining the same while that of ordinates varies. For the two curves really express the same mechanical conditions and only differ in figure, because the load lines and pole distances are drawn to different scales.

Now, in para 91, it is shown that if the pole distance be altered while the length of load line remains the same, the ordinates of the curve vary inversely with the pole distance.

And it is evident that if the pole distance remain the same while the length of load line varies, the ordinates of the curve will vary with the load line, because the inclinations of the vectors of the force polygon to one another will be greater or less according as the length of load line is greater or less.

Consequently, if x, y, be the co-ordinates of a point on a deflection curve, corresponding to pole distance k and load line h, and x, y' the co-ordinates when k becomes k' and h becomes h', then will the ratio of y to y' be as follows:—

$$\frac{y}{y'} = \frac{h \times k'}{h' \times k}$$

Now, p being the pole distance and $\frac{l}{2}$ the base of reduction of the polygon of bending moments, we have, by para. 109, for the length of the load line of the constructed curve

$$h = \frac{1}{p} \times \frac{2}{l} \times \int_o^l \mathrm{M}dx$$

with pole distance equal to $\frac{1}{3}\,l$.

For the true deflection curve we have, load line $= \int_o^l \mathrm{M}dx$, and pole distance $=$ EI.

Hence, substituting these values in the relation above obtained we

Fig. 36.

Fig. 37.

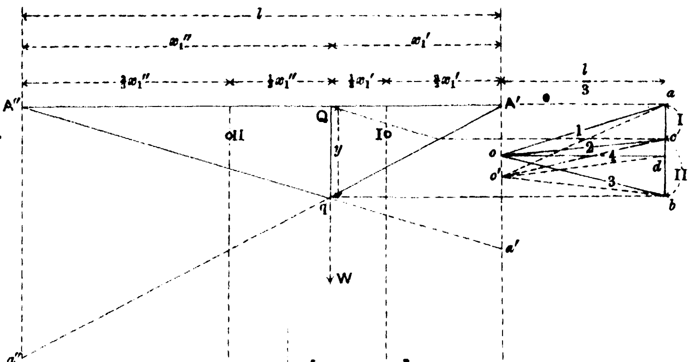


Fig. 38.

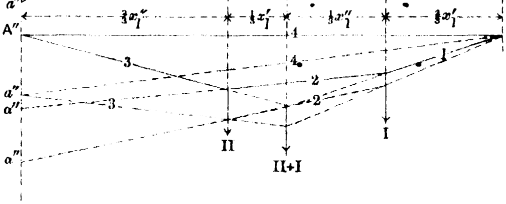


Fig. 39.

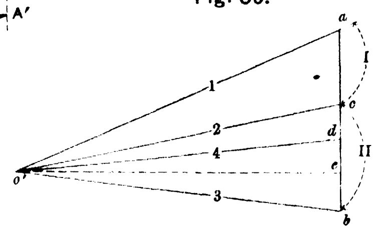
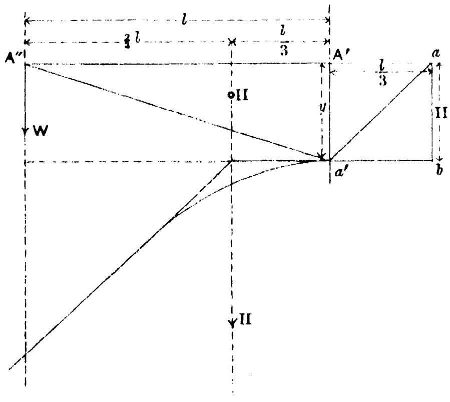


Fig. 40.



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have for the ratio of the length of the ordinate y of the true curve to that of the ordinate y' of the constructed curve, for the same abscissa x

$$y : y' :: \frac{1}{\mathrm{EI}} \int_0^l \mathrm{M}dx : \frac{1}{p} \times \frac{2}{l} \times \frac{3}{l} \int_0^l \mathrm{M}dx$$

$$\text{or } \frac{y}{y'} = \frac{pl^2}{6\,\mathrm{EI}}$$

that is, the ordinates of the constructed curve exceed those of the true curve in the ratio $6\ \mathrm{EI} : pl^2$.

E being measured in weight units and I in composite length units, p should be measured on the scale of loads and l on that of length.

Fig. 36a. Fig. 36b. Fig. 36c. Fig. 36d.

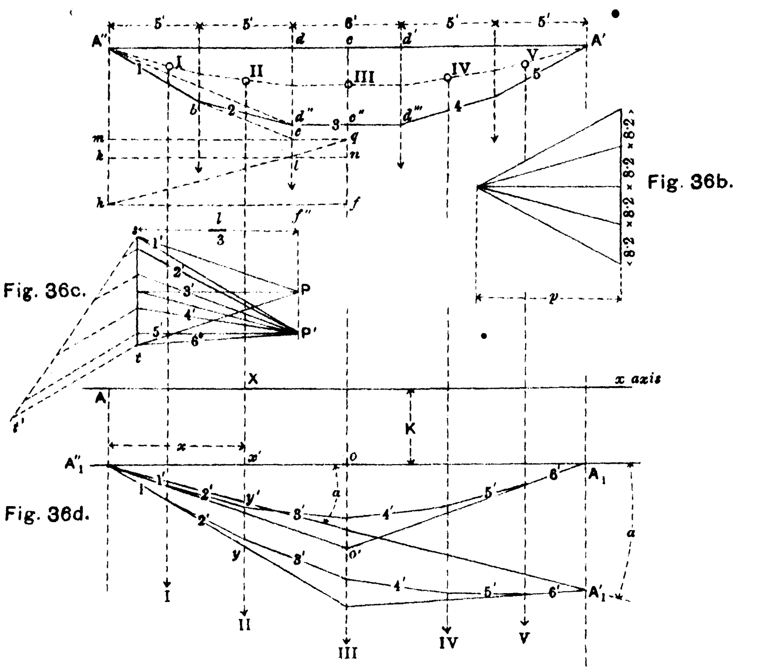
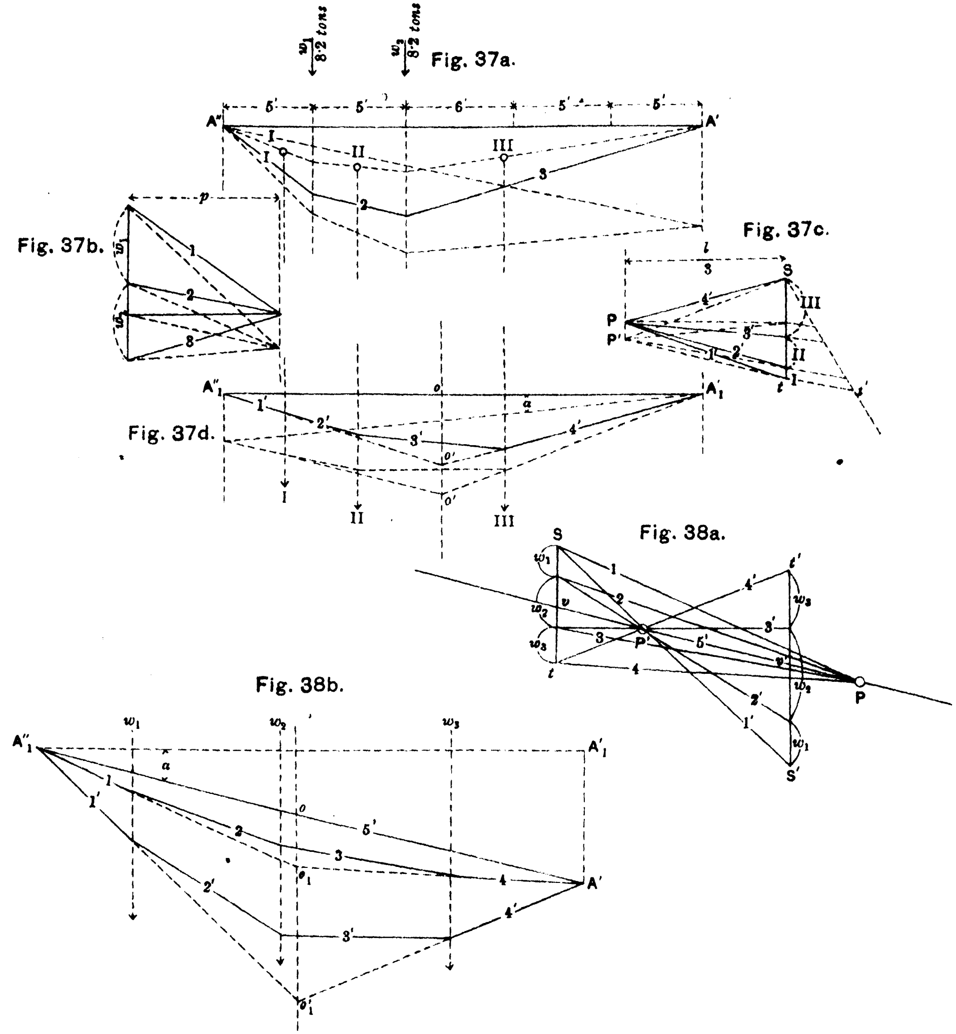


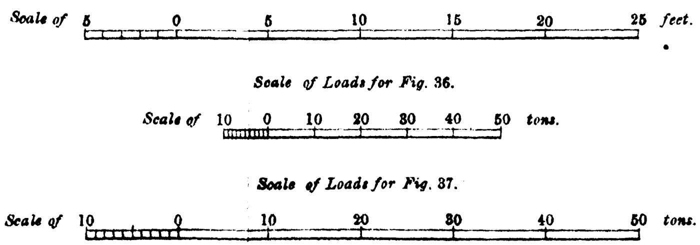
Fig. 37a. Fig. 37b. Fig. 37c. Fig. 37d. Fig. 38a. Fig. 38b.



Lineal Scale for Figs. 36 and 37.

Scale of Loads for Fig. 36.

Scale of Loads for Fig. 37.



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But, comparing the force polygons with the equilibrium polygons, we have

$$OO_1 = OA_1'' \frac{sv}{rP} \text{ and } OO_1' = OA_1'' \frac{s'v'}{r'P'}, \text{ also } \frac{sv}{s'v'} = \frac{st}{s't'}$$

Hence $$\frac{y}{y'} = \frac{st}{rP} \times \frac{r'P'}{s't'}.$$

Example II. *Figs.* 36*a* to 36*d* and 37*a* to 37*d* show the cross girder of the Plate Iron Railway Bridge designed in Example II. of Chapter XXVII., Section III., the former when both lines of rails are fully loaded, the latter when only the left pair of rails are so. The cross section of the girder is the same as that shown in *Fig.* 35*a*, *Plate* VIIA, being Example I. of the Addendum to Chapter VII.

The maximum deflection y' in both *Figs.* 36*d* and 37*d* measures about 2·7 feet on the lineal scale, but p (the pole distance of the first force polygon) in the former case measures 32 tons and in the latter about 16 tons.

We have, therefore, to substitute these quantities in the expression of para. 119, in order to determine the true deflection y, which, since it varies directly with p, will be twice as large in the former case as in the latter.

Thus, when both lines of rails are fully loaded, we have

$$y = y' \frac{pl^2}{6\,EI}$$

where $l = 26 \times 12$ inches, I (from Addendum to Chapter VII.) $= 6{,}250$ inch-pound units of inertia, $p = 32 \times 2{,}240$ lbs., and $E = 18{,}000{,}000$ lbs. per square inch.

Hence, $y = 0{\cdot}34$ inch, when both lines of rail are fully loaded, and 0·17 inch when only one pair of rails is so.

CHAPTER IX.

THE EQUILIBRIUM CURVE IN ITS APPLICATION TO FIXED AND CONTINUOUS BEAMS.

120. In Chapter VIII., a method has been explained whereby the Deflection Curve, or Elastic Line, of a beam or girder may be geometrically described. In this Chapter it is proposed to consider, in as brief a manner as possible, the question of the stability of fixed and continuous beams, the conditions of equilibrium of the two classes being similar.

121. In para. 307, Vol. I., of this Manual, a Fixed Beam is defined as "a supported beam whose extremities are so fixed that the neutral surface retains its direction at the ends under transverse load," and in para. 328 a Continuous Beam or Girder is defined as "a single beam covering several spans and resting on several supports." The Beam fixed at both ends, therefore, may be regarded, as far as its state of equilibrium goes, as being equivalent to a middle span of a Continuous Beam, the neutral surface of the end portions of which is horizontal; and a beam fixed at one end and supported at the other as equivalent to the end span of a continuous beam (para. 348, Vol. I.).

122. In para. 328, Vol. I., it is further explained that "in rigid material the pressures on the several supports (*i.e.*, the reactions of the supports) would be strictly indeterminate were there more than two of them, because the equations of equilibrium between them are only two in number;" but that "in elastic material the determination of these reactions is a perfectly definite problem for material whose elastic properties are known. The solution depends ultimately on the fundamental law of elasticity (Hooke's law, *vide* para. 91, Vol. I.) from which the equation of the elastic curve is deduced. The continuity of the beam enables the weight of the spans adjacent to any particular span to supply reactions at the two vertical end sections of the latter, which tend to reduce the deflection, and therefore also the longitudinal stress intensity, which a given load would cause on that span, were the beam

discontinuous." "The effect of the continuity is, in fact, to throw the elastic curve into a sinuous form" (*vide Fig.* 41, which shows the sinuosity greatly exaggerated) "usually convex upwards over the supports, and concave upwards near the centre of each span," (*i.e.*, near the section of greatest bending moment,) "these portions being separated by points of inflexion, of which there are commonly two in each span, so that each span is, as a rule, in the condition of a supported beam between the points of inflexion, resting on two cantilevers."

123. Perhaps the condition of the beam is more clearly described by supposing it divided as follows:—The portion lying between two points of inflexion and not including a point of support, as x_2y_3, *Figs.* 41 and 42, is in the condition of a beam loaded between, and freely supported at, the points of inflexion. The reactions at the hypothetical points of support (that is, at the actual points of inflexion) would be equal to the shearing forces acting at those points. The portion of the beam lying between two points of inflexion and including a point of support, as y_2x_2 and y_3x_3, *Figs.* 41 and 42, is in the condition of a beam loaded at its extremities and supported at the actual point of support of the continuous beam. The hypothetical load at each extremity would be measured by the actual shearing force acting there.

124. Whichever of these suppositions regarding the relative state of strain of a continuous and discontinuous beam be adopted, we are led to the conclusion that *at the points of inflexion the value of the bending moment is zero, and that at the points of support, other than the extreme ones, of the continuous beam,* (supposing the continuous beam to be freely supported and not fixed at its extremities) *the bending moment has some definite value,* and it will be seen that this conclusion is fully borne out by an examination of the equation to the elastic curve itself. For, a necessary condition in order that a curve may cut its tangent, that is, that there may be a point of inflexion, at any point is that the second differential co-efficient of the curve, $\frac{d^2y}{dx^2}$, at that point shall vanish.* Now equation (1) of para. 109 can only vanish provided $M = 0$, that is, provided the bending moment at the point in question be equal to zero.

125. We shall first examine the comparative state of strain of a continuous and discontinuous beam, and then show how the magnitude of

* Todhunter's Differential Calculus, Chap. XXI.

the moments at the points of support may be measured by means of the method explained in the last Chapter.

126. *Fig.* 41 shows roughly the form of a discontinuous beam A_1A_4, which is supported freely at its extremities A_1A_4, and also at the two intermediate points A_2, A_3, and weighted at the middle of each span by the loads W_1, W_2 and W_3, the deflection being grossly exaggerated for purposes of illustration.

Fig. 42 shows the equivalent of the beam A_1A_4, as to state of strain in discontinuous beams, and it will be observed that the single continuous beam A_1A_4 may be regarded as made up of five portions, each portion lying either between an extremity of the beam and a point of inflexion or else between two inflexion points, and each portion having its exact equivalent in a discontinuous beam.

127. If the points of inflexion be denoted by y_2, x_2, y_3 and x_3, as shown, it will be seen that the first and last portions A_1y_4 and x_3A_4 are each equivalent to a beam, unsymmetrically loaded and supported at each extremity; the middle portion x_2y_3 to a beam, supported at its extremities, and loaded with a detached weight W_2; and the intermediate portions y_2x_2 and y_3x_3, each to a beam loaded at the extremities and supported at an intermediate point (or, to two cantilevers, joined together).

If we add an intermediate portion to each extreme one, and the two intermediate portions to each extremity of the middle one, we shall obtain the equivalent arrangement shown in *Fig.* 43, and with this arrangement we propose now to deal.

128. It will be observed that the portion A_1x_2, *Fig.* 43, is equivalent to a weightless beam, supported at the points A_1 and A_2, loaded at a point midway between them, and also at the extremity x_2, which projects beyond the point of support A_2. The portion y_3A_4 is similarly loaded.

The middle portion y_2x_3 is strained similarly to a weightless beam y_2x_3, supported at the points A_2 and A_3, which are situated within its extremities, loaded at its extremities, and therefore beyond the points of support, and also loaded with a detached load W_2 hung midway between the points of support A_2 and A_3.

129. We shall confine our attention to the middle portion y_2x_3, because when the values of the moments at the points of support A_2 and A_3 have been determined, the state of equilibrium of the outer portions A_1x_2 and y_3A_4 is known also.

130. We may, further, regard the state of equilibrium of the portion $y_2 x_3$ as equivalent to what would result were the states of stress of the following two cases *superposed* the one on the other :—

(1). A weightless beam $y_2 x_3$, supported at points A_2 and A_3 within the extremities, and weighted at the extremities only, *i.e.*, beyond the points of support.

(2). A weightless beam $A_2 A_3$, supported at its extremities and loaded at its middle with detached load W_2.

131. We shall consider each case separately, and then suppose them superposed the one on the other.

Case (1) is *the reverse* of that dealt with in para. 182, Ex. 6 of Vol. I., *i.e.*, the case of a beam freely supported at its extremities, and loaded between them with two detached loads R_2, R_3. Its equilibrium polygon is, therefore, of the form shown in *Fig.* 44. The corresponding force polygon is shown in *Fig.* 45.

Case (2) is dealt with in para. 182, Ex. 4 of Vol. I. *Fig.* 47 shows the stress diagram, and *Fig.* 46 the corresponding equilibrium polygon for this case.

132. *Fig.* 48 shows the resultant polygon of bending moments for the two cases superposed. Since the bending moments of Cases (1) and (2) are of opposite sign, when the polygons are superposed, they will partially counteract one another leaving resultant moment areas, such as are exhibited by the shaded portions of *Fig.* 48 and points of zero-moment at x_2 y_3. For all sections of the beam lying between these points, the moments may be regarded as acting downwards, and for all sections beyond these points, as acting upwards.

133. The difference, however, between the conditions of the case above described, and those of a middle span of a continuous beam or girder must not be lost sight of. In the above case the weights hung at the projecting extremities y_2 and x_3 of the beam are supposed known, and consequently the moments over the points of support A_2 and A_3 are so too. The usual problem is, given certain conditions of loading over a continuous beam, required the moments over any pair of middle-span supports, and the deflection polygon, or curve, of the beam, such moments being entirely due to the loading of the contiguous spans. It will be shown that the geometrical method already explained enables these moments to be measured.

134. It will be instructive, for purposes of illustration, to apply the method first exactly as it is described in para. 109 to the case under consideration, although it will be presently shown that the objects in view can be more quickly and conveniently arrived at *without actually drawing the second force polygon at all.* This application is exhibited in *Figs.* 48, 49 and 50.

Fig. 48 shows the polygon of resultant bending moments, with the moment-areas marked I., II., III., IV., V. and VI., of which areas III. and IV., act downwards, and all the remainder upwards. Half the length y_2 A_2 has been taken as a convenient base length for reduction of the triangular moment-areas, and the resulting lengths h_1, h_2, h_3, h_4 and h_5 have been set off in their proper directions along the vertical load line *ab* of the force polygon of *Fig.* 49, so that, commencing from extremity *b*, we have $h_1 + h_2$ measured upwards to point *c*, then $h_3 + h_4$ measured downwards to point *d*, then $h_5 + h_6$ measured upwards to point *a*. Taking any pole O, with polar distance $\frac{E \times I}{p \times a}$, completing the force polygon, and describing the corresponding equilibrium polygon in the usual way so as to pass through the point of support A_2, we obtain the polygon shown by a broken line in *Fig.* 50. This polygon is then projected, by the principle described in para. 75 *q.v.*, so as to pass through the point of support A_3 (as well as A_2).

135. The example just considered illustrates in a general way, that is, with altered bases of reduction (para. 138) exactly how the tangential envelope to the elastic line or curve, of a continuous beam is drawn in. The reduction bases should, in fact, be the same as those already stated in Chapter VIII. But before the elastic curve can be drawn in, either the moments over the supports or the positions of the points of inflexion must be known. Only in the case of *hinged girders*, which will be dealt with in Section III., are the positions of the inflexion points known at once. In general, the moments over the supports have first to be determined, and then the elastic curve can, if required, be drawn in. It must be borne in mind that the two determinations are distinct, though similar in method, the one involving a question of *strength*, the other of *stiffness.*

From what has been stated in the preceding paragraphs, and also from *Figs.* 48 and 51*a* of this Volume, and *Plate* IX. of Vol. I., it will

be evident that the actual moment-areas are two in number, *viz.*—a certain upward area in the form of a trapezium, which, by joining two of its opposite angles, may be converted into two triangles, and a certain downward area, the form of which is either triangular, polygonal or curved, according to the conditions of loading; for the triangles numbered I. and VI. of *Fig.* 48 lie outside span A_1A_2. As explained in para. 132, these upward and downward component areas produce resultant areas such as those indicated by the coloured portions of *Plate* IX., Vol. I., or the shaded portions of *Figs.* 48 and 51*a* of this Volume.

For the measurement of the moments over the supports, the *component* areas are employed, the trapezium of upward moments being divided into two triangles; for the drawing in of the elastic curve, the *resultant* areas. In each case a deflection polygon is described, but only when the resultant areas are employed will the inflexion points be found to correspond with the points of zero moment.

The remainder of this Chapter will be devoted to a consideration of the measurement of the moments over the supports, the deflection polygons of which will not, therefore, be the envelopes of elastic curves. Moreover, as the case of uniformly distributed loading is that of most frequent occurrence, it alone will be dealt with in the following remarks, the question of detached loading being treated in Section III. Further, as the elastic curve has been sufficiently fully discussed in the preceding Chapter, it will not be again referred to in this Chapter.

In the remarks following, the case of a continuous girder of three unequal spans, uniformly loaded with dead and live load over the first two spans, and with dead load only over the remaining span, will be considered, the uniform load over each span producing a parabolic curve of bending moments. The remarks demand special attention, and will be made under the five headings—(1), *Moments over supports;* (2), *Reduction base and Pole distance;* (3), *Construction of "Fixed Points";* (4), *The Deflection Polygon;* (5), *Measurement of Moments at supports.*

Moments over Supports.

136. As already explained, the upward moments, which are developed at the points of support of the middle spans of a continuous girder, or the ends of a fixed one, give rise to an equilibrium polygon of the form $A_2a_2a_3A_3$, *Figs.* 44 and 48. This trapezium being divided

into two triangles by a straight line joining two corners, the upward moment-area becomes equivalent to two hypothetical forces, acting at distance $= \frac{1}{3}$ the span from either support, that is, through the centre of gravity of the representative triangle, each force being represented and measured by $\frac{1}{2}$ span $\times$ base of representative triangle. For the same span, therefore, each force is proportional in magnitude to the length of that base. Thus in *Fig.* 51*a*, the forces figured III. and V. are proportional to the base-lengths A_2a_2 and A_3a_3 the representative triangles $A_2a_2A_3$ and $a_2a_3A_3$ being of the same height l_2. But for two contiguous spans, since the two representative triangles have a common base-length, the forces are proportional in magnitude to the lengths of the spans. Thus the forces figured II. and III. of *Fig.* 51*a* are proportional in magnitude to l_1 and l_2, being represented by the area of the triangles $A_1A_2a_2$, and $A_2a_2A_3$, which stand on common base A_2a_2.

137. On each side of the middle support of two contiguous spans, therefore, or of the section of fixation of a fixed beam, there is a hypothetical force acting upwards at a distance of $\frac{1}{3}$ the corresponding span from the support or section of fixation. These two upward forces have a single resultant acting at a distance from the line of action of each component which is inversely proportional to the magnitude of the component, that is, to the length of the corresponding span, and if these proportionate lengths (being together equal to $\frac{1}{3}$ the sum of the spans) be set off from the line of action of each component force, the line of action of the resultant is determined. Thus, in *Fig.* 51*a*, the upward force figured II. acts at distance $= \frac{1}{3}l_1$ to left of middle support A_2, and that figured III. at $\frac{1}{3}l_2$ to right of it; hence, the line of action of the resultant force, II. + III., will divide the total distance $\frac{1}{3}(l_1 + l_2)$ between the lines of action of the component forces II. and III., in a ratio which is inversely proportional to the respective magnitudes of those forces, that is, it will be distant $\frac{1}{3}l_2$ from the line of action of II., and $\frac{1}{3}l_1$ from that of III. Similarly for the other middle point of support A_3.

Reduction base and Pole distance.

138. In the example of para. 134, half the length y_2A_2, *Fig.* 48, has been taken as the base of reduction for moment areas, and $\frac{E \times I}{p \times a}$ as the pole distance for the second force polygon. As already stated, the

base of reduction will always in subsequent examples be taken equal to half the length of span of a fixed beam, or of a selected span of a continuous beam; and the pole distance of the second equilibrium polygon at $\frac{1}{3}$ the actual span of a fixed beam, or selected span of a continuous beam.

139. The span of the continuous beam usually selected for reduction is that which furnishes the greatest bending moment. It becomes necessary, therefore, to make a reduction for other spans. This may be done as follows:—

140. Let l be the selected span, giving greatest bending moment y, l' the length, y' the greatest bending moment, of any other span; supposing the spans to be uniformly loaded (the most general case), we have the parabolic moment area of span $l' = \frac{2}{3} y'l'$ which, reduced to base $\frac{l}{2}$, is equal to a rectangle of height $\frac{4}{3} y' \frac{l'}{l}$. The intercept cut off from the vertical drawn through a point of support by this hypothetical force, supposed hung at the centre of the girder, is given by the relation (*vide* para. 117),

Pole distance : half span : : force : intercept

for the triangle formed by the two extreme sides, produced if necessary, of the polygon of downward moments, (or the tangents at the supports in the case of a curve,) and the vertical through the point of support, is similar to the triangle formed by the force polygon; hence, since pole-distance $= \frac{1}{3} l$, and force $= \frac{4}{3} y' \frac{l'}{l}$, and half span $= \frac{l'}{2}$, we have required intercept $= 2y' \left(\frac{l'}{l}\right)^2$. The ratio $\left(\frac{l'}{l}\right)^2$ may be termed the *reduction factor*, to be applied to all spans, other than the standard one, and the multiplication by this ratio may be easily performed graphically,* as follows:—

Along the same straight line AB, make AB $= l^2$, and AC $= l'^2$ (*Fig.* 52), set off AD $= y'$, the ordinate representing the maximum bending moment at middle of span l', at any convenient angle with AB, and join DB. A straight line drawn through C parallel to BD to meet AD in E, gives AE $=$ AD $\frac{AC}{AB} = y' \left(\frac{l'}{l}\right)^2$.

* Dubois' "Graphical Statics," p. 132.

In the example shown in *Fig.* 51, this reduction is supposed to have been made in the case of the first and third spans, the middle one, A_2A_3 being taken as the standard span.

141. For the standard span, the length of the intercept is obviously equal to $2 \times y$, that is, to twice the length of the maximum ordinate of the curve of bending moments—a very convenient quantity.

142. Thus, the intercepts on the verticals through the points of support are completely determined, and since their lengths depend upon the data of the problem, they can be found at once, either by calculation or diagram, and laid off on those verticals, as shown in *Fig.* 51*b*, and it is to be particularly noted that by joining the extremities of these intercepts, as in *Fig.* 51*b*, we *are enabled to determine the intercepts on any other vertical whatever, that is, to measure the moments of the downward forces about any chosen vertical, all on the same scale, that is, with the same base of reduction.*

Construction of the "Fixed Points" and of the Deflection Polygon.

143. The following data are now available for drawing in the Deflection Polygon, *viz.*, (1), *The positions of the points of support*; (2), *The lines of action of the hypothetical forces*; (3), *The lengths of the intercepts made on any vertical by those of the forces which act downwards.* We proceed to explain how certain other points, lying in the sides of the polygon, may be determined; and as their positions depend upon the following *purely geometrical* relation, it may as well be stated and proved forthwith.

144. *Theorem.—If the three angles of a triangle* f_1, b_1, e_2, *Fig.* 51*c*, *always travel along three fixed straight lines, which are parallel to one another, whilst two of its sides* f_1b_1, f_1e_2 *pivot round the fixed points* a_1' *and* A_2 *respectively, then will the third side* b_1e_2 *pivot round a fixed point* d_2, *in the same straight line with* a_1' *and* A_2.

The Theorem is obviously in no way altered by supposing the points A_2 and b_1 to lie on the same vertical, which they actually do when the spans are equal. As the proof is simplified under this supposition, we shall confine ourselves to the case shown in *Fig.* 53, when $l_2 = l_1 = l$.

The following is a simple analytical proof, the co-ordinate axes being supposed rectangular, the origin to coincide with the point of support A_2, and the x-axis with the line of supports $A\ A_2$.

Putting the known intercept $A_1a'_1 = a$, and supposing any straight line $a_1'b_1$ to be drawn from a_1' so as to intercept a length $A_2b_1 = s$ on the y axis, we have for the corresponding intercept t on the x-axis

$$t = s\frac{l-t}{a}$$

$$\therefore t = \frac{sl}{a+s}$$

and the equation to $a_1'b_1$ in terms of the intercepts on the co-ordinate axes, is *

$$-\left(\frac{a+s}{l}\right)x + y = s.$$

If $x = -\frac{l}{3}$, then $y = \frac{2s-a}{3} = f_1f = ee_2$, so that the equation to the straight line b_1e_2, joining the points b_1 (O, s), and $e_2\left(\frac{l}{3}, -\frac{2s-a}{3}\right)$ is

$$y - s = \frac{\frac{-2s+a}{3} - s}{\frac{l}{3}}x$$

$$\therefore y = -\left(\frac{5s-a}{l}\right)x + s, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to the straight line a_1' A_2 is evidently

$$y = \frac{a}{l}x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, equating the values of y in equations (1) and (2) we have

$$\frac{a}{l}x = \left(\frac{a-5s}{l}\right)x + s$$

or

$$x = \frac{l}{5}$$

which is evidently independent of s. Similarly $y = \frac{a}{5}$.

Hence, for given values of a and l, the point d_2 is *fixed.*

145. If the more general case shown in *Fig.* 51c be treated in a similar manner, that is, the case where the lengths of contiguous spans measure l_1 and l_2, and the horizontal distance between the point of support A_2 and the vertical through b_1 is, therefore, equal to $\frac{1}{3}$ ($l_2 \backsim l_1$), the following expressions will be obtained for the values of x and y giving the "fixed" point lying in span l_2,

$$x = \frac{l_2^2}{2l_1 + 3l_2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$y = \frac{a}{l_1}\left(\frac{l_2^2}{2l_1 + 3l_2}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

* Todhunter's Conic Sections, 4th Edition, p. 14.

Hence, for a given position of a_1', that is, for a given length $A_1 a_1'$, the point d_2 is *fixed.*

146. It will be observed that *the value of x is entirely independent of the loading,* while that of y varies with the position of a_1'. Now the length of the intercept A_1a_1' varies directly with the magnitude of the maximum bending moment $\frac{wl^2}{8}$, where w is the intensity of uniform loading and l the length of the span, and inversely as the pole distances of the equilibrium curves, for it represents on a certain scale of ordinates the moment of the area of the curve of bending moments about the vertical through A_1. If, then, the scale of ordinates of the deflection curve be changed, that is, if the pole distance be changed, or in other words, if the curve be projected as explained in para. 50, the length A_1a_1' will also change, but as the point a_1' always lies in the vertical through A_1, so also will the corresponding "fixed" point always lie on the vertical through d_2.

Hence, *given the lengths l_1 and l_2 in the above equations, the value of x is constant for all deflection curves whatever drawn for the given system of loads, and the vertical drawn at distance x from the point of support represents the locus of the "fixed" points of the several curves.*

147. If the position of the first "fixed" point be determined in the manner above explained, that of a second one may be found in exactly the same manner, since the moment of the downward load about the vertical drawn through the first "fixed" point, that is, the length of the intercept on the vertical drawn through that point, is known by the construction described in para. 142. In this manner the whole series of "fixed" points of a continuous girder may be determined, commencing from each end, supposed supported, at which the bending moment is zero. In each middle span two "fixed" points, and in each end one one "fixed" point, may thus be determined.

148. It will be observed that no hypothesis is made in the above method as to any difference in height of the supports, and the results are, therefore, true when these are not on a level. But it must be carefully borne in mind that if *the "fixed" points be set out on the assumption that the supports all lie in a straight line, not necessarily horizontal, then, when erecting the girder, the greatest care must be taken to set the surfaces of the bed plates in a plane fulfilling these conditions.*

149. The "fixed" points of a beam fixed at both ends may be determined by supposing the neutral surface of the beam extended at either end, a distance equal to the length of the span, and these hypothetical spans to be so loaded as to render the neutral surface horizontal at the sections of fixation. The conditions of equilibrium are thus reduced to those of the middle span of a continuous beam of three equal spans, the middle one of which is loaded in the given manner, and the outer ones hypothetically.

150. Similarly, a beam fixed at one end and supported at the other may be regarded as the span, whose loading is known, of a continuous girder of two equal spans, which is supported at its extremities and at a point midway between them, the hypothetical span being so loaded as to render the neutral surface horizontal at the section of fixation.

151. A continuous beam, fixed at its extremities, may be treated as one supported at its extremities, by supposing the neutral surface extended at each end, a distance equal to the end span, and then supposing these hypothetical spans to be loaded in such a way as to render the neutral surface horizontal at the supports.

152. When the weight of the girder is disregarded, the spans are considered *unloaded*, being subjected only to the upward or downward moment, due to the effect of a load hung elsewhere on the girder. Since in such spans the downward applied load is nil, the intercepts on the verticals through the points of support are so too, (corresponding, that is, to the lengths a_1a_1' and d_2d_2' of *Fig* 51*b* or A_1a_1' and d_2d_2' of *Fig.* 51*c*) so that the vectors, such as $a_1'b_1$ or $d_2'b_2$ of *Fig* 51*c* must be drawn from the actual point of support or "fixed" point itself, as the case may be, as in *Fig.* 54. *Fig.* 54 shows the construction for determining the "fixed" points in a series of unloaded spans. The process may be briefly described as follows:—

153. The lines of action of the resultants of the upward loads acting either side the supports having been drawn in, as described in para. 137 (shown as B_1 and B_2 in *Fig.* 51*a*, and b_1, b_2, &c., in *Fig.* 54), take *any* point as b_1 in the resultant line, corresponding to the second support A_2, *Fig.* 54, and join A_1b_1 cutting the upward load line to left of A_2 in f_1. Join fA_3, and produce it to meet the upward load line to right of A_2 in e_2. Join b_1e_2 cutting the line joining A_2A_3 in d_2. The point d_2 is that in which the "fixed" vertical meets A_2A_3

Now treat d_2 as an end support (*i.e*, similarly to A_1) and taking any point b_2 in the line of resultant upward loading at A_3, join d_2b_2 and treat the portion d_2A_3 exactly as A_1A_2 was treated. In this way determine the "fixed" vertical in span A_2A_3. Repeat this process until the last span A_6A_7 is reached, and then commencing at A_7 work backwards to span A_2A_1 in exactly the same manner. In this way one "fixed" vertical will be obtained in each end span, and two in each intermediate one.

154. It will thus be seen that since, for given lengths of span, the horizontal distance of all "fixed" points from a given support is the same, that is, the value of x in equation (3) of para. 145 is constant for *loaded* spans, the "fixed" verticals can be first laid out as in *Fig.* 54, on the hypothesis that the spans are unloaded, and the intercepts made on them by the sides of the polygon of the downward load afterwards.

155. In *Fig.* 55, five of the six spans shown in *Fig.* 54 are supposed unloaded, the span A_4A_5 only being loaded, and it will be seen that, in order that the deflection polygon may always pass through a point of support, the construction demands that the portion of it lying to the left of the loaded span shall pass through the "fixed" points that fall in the left outer third of each span, and the portion to the right of the loaded span through the "fixed" points lying in the right outer third of each span, also that these "fixed" points occur where the curve crosses its tangent, that is, at inflexion points. It will also be seen that in this case the moments at the supports must be alternately positive and negative, and increase from the end of the girder towards the loaded span.

To describe the Deflection Polygon of a Continuous Girder.

156. The Deflection Polygon of *Fig.* 51*e* may now be drawn in.

The following are the successive steps:—

Step I. Set out the "fixed" verticals on the hypothesis that *the spans are unloaded*, as explained in para. 153, and draw verticals through the points of support, as A_1a_1', d_1d_1', d_2d_2', &c., of *Figs.* 51*b* and *e*.

Step II. Determine the actual "fixed" points, *the spans being loaded*, as follows:—As shown in *Fig.* 51*c*, set off on vertical through A_1 downwards, the intercept A_1a_1' (as given by *Fig.* 51*b*); join $a_1'A_2$ and pro-

duce it to meet "fixed" vertical to right of A_2 in d_2. By para. 144, d_2 is required "fixed" point. From d_2 set off the intercept d_2d_2' (from *Fig.* 51*b*), join d_2' with A_3 and produce to meet "fixed" vertical to right of A_3 in d_4, which is required "fixed" point. From d_4 set off the intercept d_4d_4' (from *Fig.* 51*b*); and join $d_4'A_4$. Then, commencing from A_4 go through a similar process working back to A_1, setting off the intercept A_4a_4', joining $a_4'A_3$ and thus "fixing" d_3. In this way, as before, "fix" the points d_3', d_1, d_1', and join $d_1'A_1$ (*Fig.* 51*d*).

Step III. Join $a_1'd_1$ cutting A_1d_1' in C_1 and the line of action of Force II. in *f*. Join $d_3'd_2$ cutting Force III. in *e*. Produce $a_1'd_1$ and $d_3'd_2$ to meet in D_2 (*Fig.* 51*e*).

In a similar way join $d_2'd_3$, cutting Force V. in *f*, and $a_4'd_4$, cutting Force VI. in *e*, and produce to meet $a_2'd_3$ in D_3.

Let $d_3'd_2$ and $d_2'd_3$ intersect in C_2 and $a_4'd_4$ and A_4d_4' in C_3.

Step V. Checks.—The points C_1, C_2 and C_3 must lie respectively on the lines of action of the downward applied loads, *i.e.*, on those of Forces I., IV. and VII., and the points D_2 and D_3 on those of the resultant upward loads, *i.e.*, on Resultants II. and III., V. and VI., respectively. Also, the straight lines *fe* must, in each case, pass through a point of support.

157. In dealing with polygons for fixed beams the so-called "fixed" point near the section of fixation becomes the extremity of the side of the polygon passing through that point of support, distant, that is, $\frac{1}{3}$ the span from the section of fixation. From this point, therefore, the intercept made by the downward load on the vertical passing through that point would be measured.

Measurement of Moments at the Supports.

158. By the "Moments at the Supports" is meant the moments of the upward forces (para. 132). If, then, the sides C_2e and C_2f of span A_2A_3 of *Fig.* 51*e* be produced to meet the verticals through the supports A_2 and A_3 in the points a_2 and a_3, then will the lengths of the intercepts A_2a_2 and A_3a_3, *provided the span* A_2A_3 *be the standard one*, measure the moments of the upward forces at the supports A_2 and A_3 on the same scale as the ordinates of the parabolæ of *Fig.* 51*a* represent the bending moments at successive vertical sections of the girder. For, if through the point *e* in span A_2A_3, *Fig.* 51*e*, the straight line *eβ* be

drawn parallel to C_2D_3 meeting the vertical through A_2 in β, then will the sides of the triangle $a_2e\beta$ be parallel to those of the second force polygon; and since the distance of e from $a_2\beta$ is equal to $\frac{1}{3}$ A_2A_3, *i.e.*, the pole distance, the triangle is in every respect equal to the second force polygon, and $a_2\beta$ therefore represents in magnitude the hypothetical load at C_2, (*i.e.*, the area of the polygon of bending moments, with base of half the span) and a_2A_2 that at e, (*i.e.*, the area of the triangle $A_2a_2A_1$, *Fig.* 51*a* with same base). Similarly it may be shown that the lengths A_1a_3 of *Figs.* 51*a* and 51*e* are identical.

Hence the values of A_2a_2 and A_1a_1 are completely determined.

For any other span than the standard one, the intercepts of *Fig.* 51*e* must be multiplied by the square of the ratio of the spans as explained in para. 140.

159. In the above investigation, the case of uniform loading has been dealt with. The case of detached loading is considered in Section III. For a more detailed treatment of the subject, Students are referred to *Dubois' Graphical Statics*, or *Chalmers' Graphical Determination of Forces in Engineering Structures.*

160. The contents of this Chapter should be compared with those of Chapters XVII. and XVIII. of Vol. I., in which the subject is treated by Analytical Methods.

In these Chapters of Vol. I. the subject of the measurement of the Shearing Stress developed in Continuous and Fixed Beams is fully dealt with; this will be again referred to in Section III.

Fig. 41.

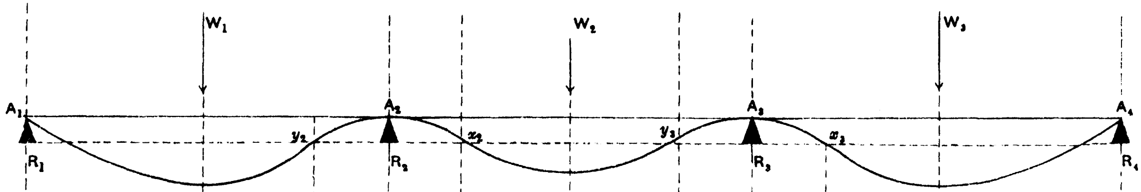

Fig. 42.

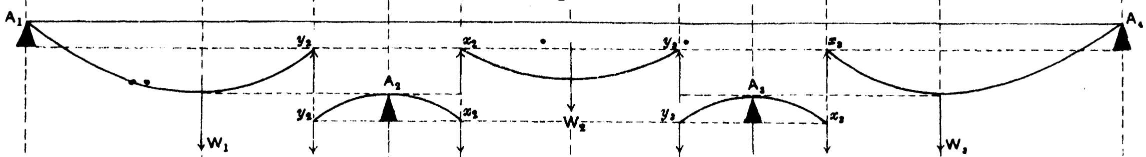

Fig. 43.

No. II

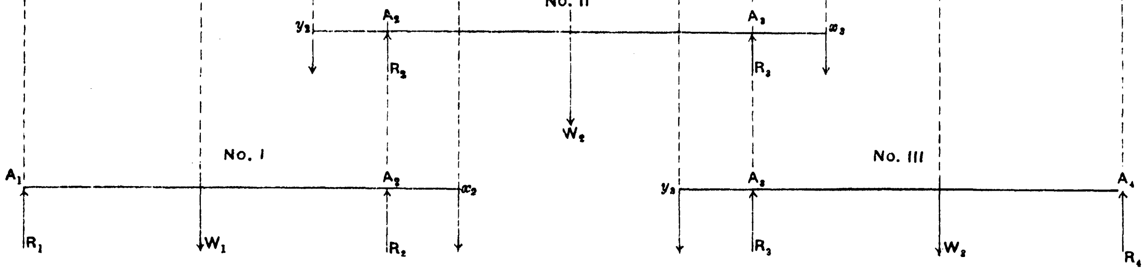

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Fig. 44.

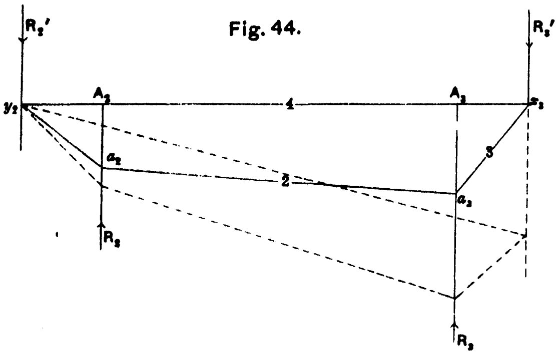

Fig. 45.

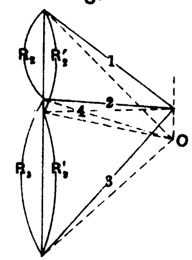

Fig. 46.

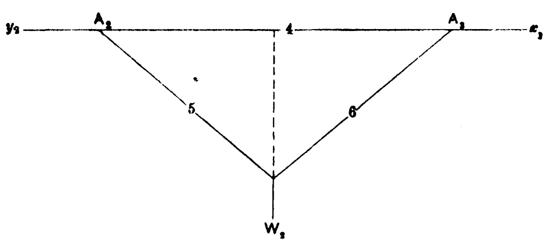

Fig. 47.

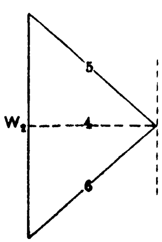

Fig. 48.

Fig. 50.

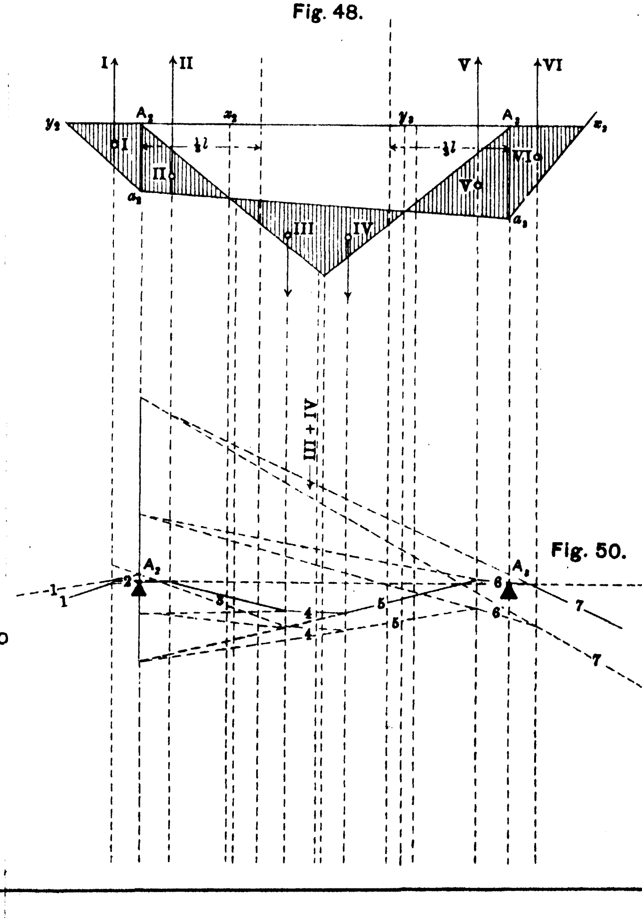

Fig. 49.

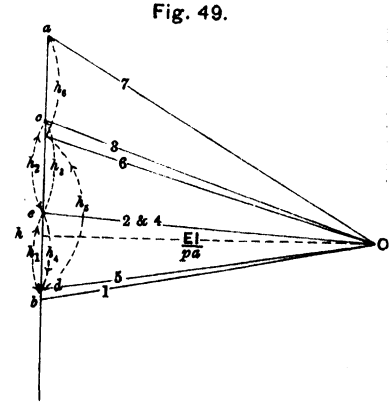

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Fig. 51a.

Fig. 51b.

Fig. 51c.

Fig. 51d.

Fig. 51e.

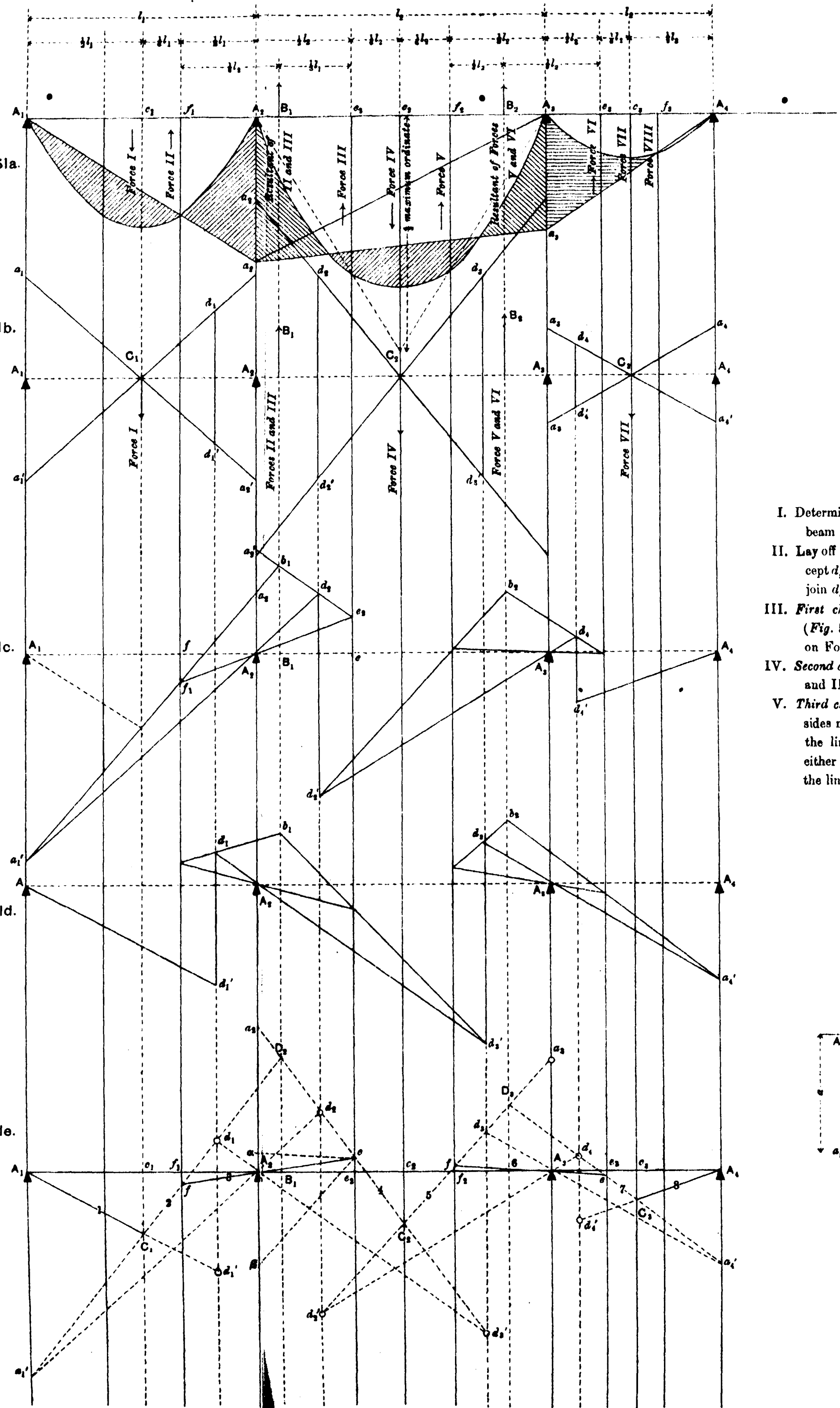
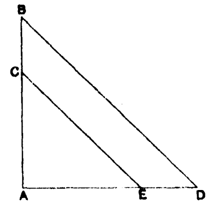


Fig. 52.



N.B.—*Hints for drawing Fig.* 51*e*.

I. Determine verticals through "fixed" points on hypothesis that beam is unloaded (para. 155).

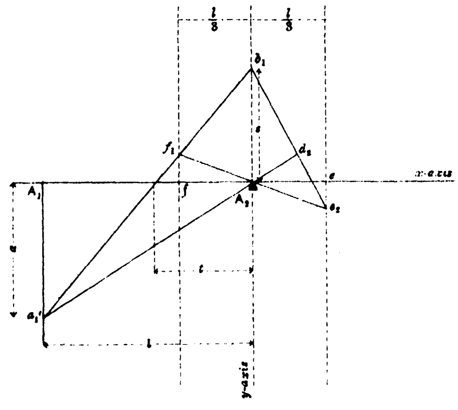
II. Lay off intercept A_1a_1, join a_1A_2 and produce to d_2; lay off intercept d_2d_2', join $d_2'A_3$ and produce to d_4; lay off intercept d_4d_4' and join $d_4'A_4$. Repeat operation commencing from extremity A_4.

III. *First check.* A_1d_1' and $a_1'd_1$ (*Fig.* 51*e*) intersect on Force I. (*Fig.* 51*a*); d_2d_3' and $d_2'd_3$ on Force IV., and d_4a_4' and $d_4'A_4$ on Force VII.

IV. *Second check.* $a_1'd_1$ and $d_2'd_2$ intersect on resultant of Forces II. and III., and $d_3'd_3$ and $a_4'd_4$ on that of Forces V. and VI.

V. *Third check.* The line joining the intersections of the pair of sides meeting on a resultant force line above any support with the lines of action of the upward component loads acting either side that support, should pass through the support (*e.g.*, the lines *fe* of *Fig.* 51*e* must pass through point of support).

Fig 53.



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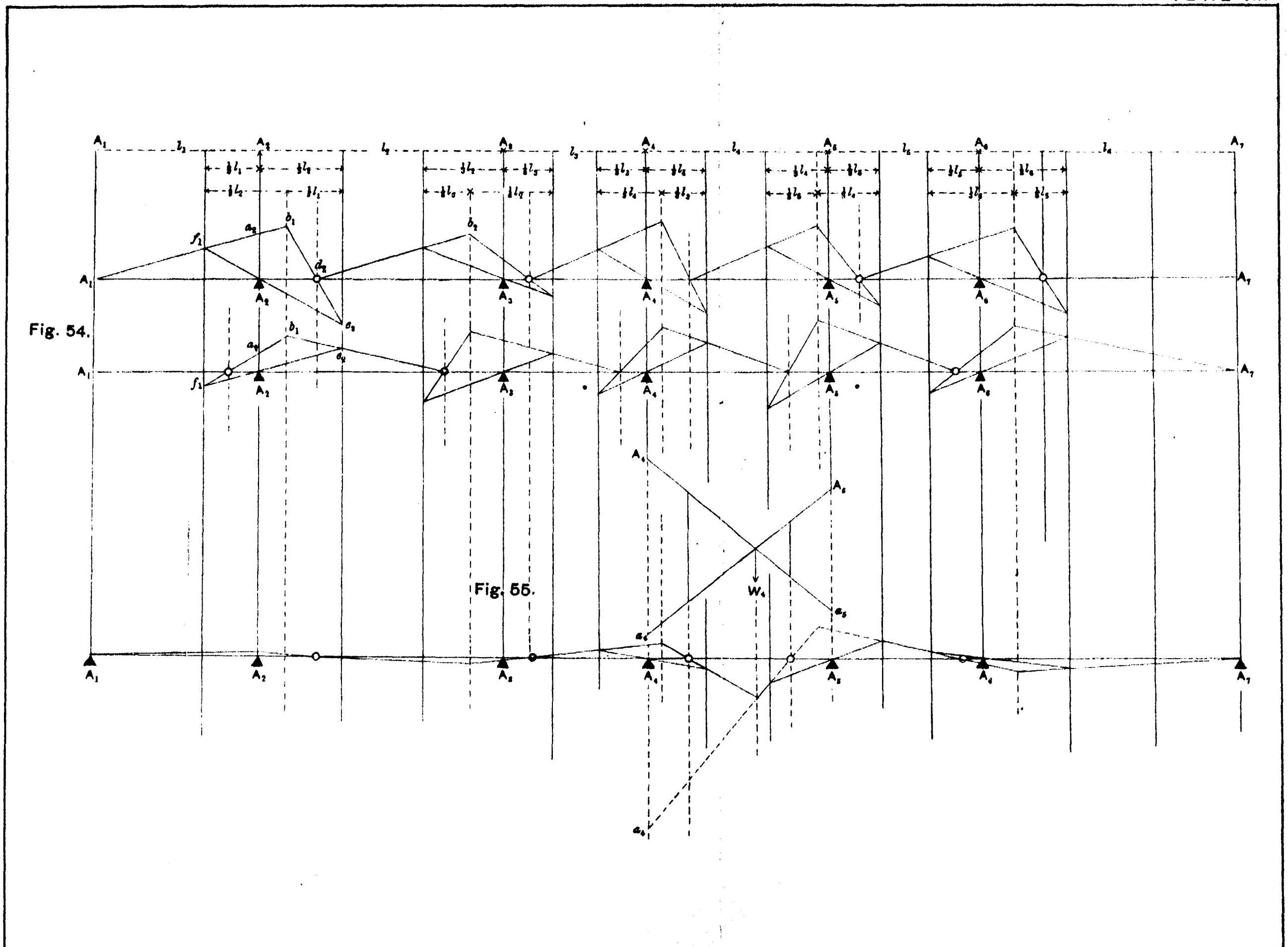

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ADDENDUM TO CHAPTER IX.

It will be observed that the demonstration given in para. 109 depends on a geometrical, and not on a mechanical, property.

It would seem superfluous to remind the Student that the results of observations and experiments in Physical and Engineering Science are, as a rule, more conveniently expressed by curves, plotted graphically to scale at once, than by equations; and likewise also the successive differentiation and integration of such curves.

Any plane curve, then, whose equation $y = f(x)$ would be an explicit function of x, having been thus plotted, its differentiation, corresponding to its first derived equation $y = f'(x)$ and expressing the rate of change of ordinate in relation to abscissa, may be readily plotted as follows:—

Draw tangents to the curve $y = f(x)$ at a series of chosen points, and through *any* convenient pole draw a series of vectors parallel to these several tangents. Draw also through the pole a vector parallel to the X-axis.

Now draw at unit distance from the assumed pole a straight line parallel to the Y-axis and cutting the vectors. The distances from the point in which this straight line cuts the vector, which is parallel to the X-axis, to the several points in which it cuts those which are parallel to the tangents give a series of lengths of ordinates of the first derived curve $y = f'(x)$, which correspond to the abscissæ of the several points on the original curve at which tangents were drawn.

For, if a be the inclination to the X-axis of any one of these tangents, and h the corresponding distance, or length of ordinate, above referred to, then at that point on the curve we shall have

$$\frac{dy}{dx} = \tan a = \frac{h}{1}.$$

If, moreover, a pole distance greater than unity be employed, obviously the length of ordinate will be correspondingly greater.

In a similar manner this first derived curve might be differentiated,

and thus the second derived curve, corresponding to the equation $y = f''(x)$ and expressing the rate of change of ordinate in relation to abscissa of the first derived equation, obtained. Similarly the third derived curve, and so on, might be described.

From the above explanation the process of graphical integration, as exemplified in the case of the elastic curve, will be readily understood.

PART II.

SECTION I.—BLOCKWORK STRUCTURES AND CRANES.

CHAPTER X.

BLOCKWORK STRUCTURES.

161. Of structures not employed in spanning an interval, those alone which directly resist the applied load at their own points of support are exempt from transverse strain. Structures of Uncemented Blockwork, the material of which is incapable of resisting a tensile strain, and therefore of balancing a resultant load (including of course the weight of the structure) obliquely applied, producing pure transverse strain, cannot be subjected to the action of a resultant bending moment.

Cantilevers, on the other hand, and structures of Cemented Blockwork, the materials of which are capable of resisting both compressive and tensile strain, and therefore of resisting pure transverse strain, may be subjected to the action of a total resultant load obliquely applied, giving rise to shearing stress and resultant bending moments.

As certain conditions of stability are common to structures of uncemented and cemented blockwork alike, the two classes will be considered together as far as possible.

162. Blockwork Structures are composed of a number of pieces having plane surfaces of contact, such as bricks or blocks of stone, which are generally cemented together with mortar.

It will be well here to draw the student's attention to the two distinct meanings of the term "joint" often indifferently applied. It is used to mean the "masonry or mortar joint," otherwise called the "bed joint,"

separating the actual surfaces of two contiguous blocks; it is also used to mean the surface of a plane, described in any particular direction for a particular purpose, and dividing the material of the block into two portions. Such a joint would be more correctly distinguished as a "hypothetical joint." For instance, in measuring the stability of an arch ring and its superstructure, it is convenient to employ hypothetical "joints" dividing the structure vertically and at equal horizontal distances apart in order to estimate the vertical load, but these hypothetical "joints" must in no way be confounded with the actual "bed joints" of the voussoirs, which are not vertical but inclined planes. Sliding, therefore, will not occur along the former, but along the latter joints; crushing too, if any, is most likely to occur at an extremity of a bed joint.

163. All heavy structures (such as abutments of arches or roof frames, retaining walls for earth or water, tall chimneys liable to be shaken by the wind, &c.,) in which, even with the greatest care, unequal settlements and consequent dislocation of joints or even fracture of the blocks themselves are likely to occur, are designed on the hypothesis that the function of the mortar is merely to fill the interstices of the bed joints, and in no way add to the stiffness of the structure. Such structures, in fact, are made to depend for their stability on their weight and form alone, and are treated as *Structures of Uncemented Blocks.*

164. In designing lighter structures, on the other hand, such as long enclosure walls, where the probability of unequal settlement, or of being shaken by the wind while the mortar is still green, is small, the tenacity and adhesion of the mortar must be taken into account, because, "unless the action of the wind on long walls is very much overrated, the majority of long enclosure walls actually constructed and standing must frequently depend for their stability on the tenacity and adhesion of their material, their weight and form alone being insufficient to give them power to resist the pressure of the storms which occur every year, * * * and which must act at right angles to the face of the wall."*

Structures of Uncemented Blockwork.

165. It is obvious that a structure of uncemented blocks may fail in three ways—

(1). By the overturning of some one or more blocks on edge.

* Cols. Wray and Seddons' "Instruction in Construction," 3rd Edition, p. 321.

(2). By the sliding of some one or more blocks on the surfaces of contact with the adjoining blocks, *i. e.*, their bed joints.

(3). By crushing of the material at a joint, if the pressure be excessive or distributed over too small an area; and this would probably always take place, except in very light structures of strong material, when overturning is just about to take place at a bed joint, as crushing would be most likely to occur somewhere along an edge of the block.

166. For, consider the equilibrium of any one block such as CDEF, as shown in *Fig.* 56, *Plate* XIII., which is acted on by its own weight W, through its centre of gravity G, and by the impressed forces P_1, P_2, P_3, and P_4, acting as shown. In order to determine the magnitude and line of action of the resultant force R, describe a stress diagram, *abcdef*, as shown in *Fig.* 57, the side *ab* of which represents P_1; *bc*, P_2; *cd*, P_3; and *ef*, W; take *a* as pole, and join *ac*, *ad*, *ae*, and *af*. Then *af* represents R. The line of action of the resultant of P_1 and P_2 is determined by drawing through the point of intersection of the lines of action of P_1 and P_2 a straight line $(P_1 + P_2)$ parallel to *ac*; that of $(P_1 + P_2 + P_3)$ by drawing through the point of intersection of the lines of action of $(P_1 + P_2)$ and P_3 a straight line $(P_1 + P_2 + P_3)$ parallel to *ad*; and so on. In this way the point O in which the resultant $R = (P_1 + P_2 + P_3 + P_4 + W)$ of the whole system meets the surface AB of the block on which CDEF rests is found. O is the centre of pressure, or centre of resistance (para. 5) of the joint AB.

167. Now, it is obvious from statical considerations that O must fall within EF, resistance being afforded only between the points E and F of the bed joint, otherwise tilting of the block will ensue.

168. It is also evident that the direction of R must be such that its inclination to the normal at O must not exceed the angle of friction of the material of the blocks, otherwise motion along EF will ensue. In other words, the angle RON must not exceed the angle of friction, or, as it is usually called, the angle of repose of the material.

"It would appear desirable to allow a margin of security to guard against accidents and the disturbing effects of shocks, and only to consider a structure safe in which the direction of the resultant forces on any bed joint makes with the normal to that joint a greater angle than that of which a certain fraction of the co-efficient of friction is the tan-

gent, and it is proposed to fix this fraction at $\frac{4}{5}$. There is seldom any difficulty in doing this, as, in cases where sliding is at all likely to occur, the bed joints can be inclined so as to reduce the angle."*

169. In structures subjected to a single oblique external pressure applied at or near the top, as in the case of house walls and abutments of arches, it is evident that the direction of the resultant thrust makes a greater angle with the normal to the bed joint (supposing the bed joints to be parallel with one another) at the joint next below the point of application of the thrust than at any other, and it is, therefore, only necessary to enquire into the stability of that joint.

170. If, however, more than one oblique pressure be applied, it may be necessary to enquire into the stability of several of the bed joints as regards sliding.

171. If the centre of pressure at any bed joint were to be actually at the edge of that joint, the whole pressure would be concentrated on that edge, which would, in any ordinarily heavy structure, be unable to bear the pressure, and would therefore be crushed. If the centre of pressure were even too near to the edge of the joint, there might be sufficient concentration of the pressure on the edge to crush or crack the material, and failure might ensue. It becomes necessary, therefore, to consider how far inside the outer edge the centre of pressure must be, in order to eliminate the risk of crushing.

172. But before proceeding to the question of strength at a Plane Joint, which involves the consideration of the distribution of stress over the plane surface, the principles of which are applicable alike to structures of Cemented and Uncemented Blockwork, we may at once state the *General Conditions of Stability of a Structure of Uncemented Blocks.*

The conditions of stability of a structure of uncemented blockwork are two in number, viz.:—

I.—*The line of resistance must intersect every bed joint sufficiently far within the outer edges to prevent any risk of their crushing.*

II.—*The angle which the resultant pressure on any bed joint makes with a normal at that joint must not exceed $\frac{4}{5}$ of the angle of repose (i.e., that angle the tangent of which is $\frac{4}{5}$ of the co-efficient of friction).*

It will be observed that in these conditions no account is taken of

* Wray and Seddons' "Instruction in Construction," 3rd Edition, page 333.

the *tensile* strength of the materials, their power to resist crushing and sliding being alone relied on.

173. In *Figs.* 58 and 59, *Plate* XIV., the treatment of a general case of a structure of uncemented blockwork is shown. Each block is supposed to be subjected to the action of its own weight, and a single load only. The several weights W_1, W_2, W_3, W_4 act through the centres of gravity of the respective blocks I., II., III., and IV., and the applied loads are represented by P_1, P_2, P_3, and P_4,—P_1 being applied to block I., P_2 to block II., and so on.

A polygon of external loads (including the weights) is first drawn (*Fig.* 59), and then the points a_1', a_2', a_3', a_4', in which the total resultant load on each block meets the upper surface of the block below it are determined in the usual way. The polygon formed by joining the points of resistance, or centres of pressure a_1', a_2', a_3', a_4', is known as the *polygon of resistance.* It will be observed that if the thickness of each block is supposed to become infinitesimally small, the polygon becomes a curve. It will also be observed that the polygon of resultant active pressure does not coincide with the so-called polygon of resistance; the *curves* of active and passive action will, however, so coincide (para. 52).

The *line of resistance* of a structure, as defined in para. 5 is, in fact, *the locus of the centres of pressure or resistance of successive joints*, the centres of action and reaction being coincident and the planes of the joints parallel to one another. The direction of the resultant pressure at any joint does not, therefore, necessarily coincide with that of the tangent to the line or curve of resistance at the joint's centre. For instance, in Chap. XV., it is shown that the line of resistance of a reservoir wall or dam, when empty of water, is inclined to the vertical, although the resultant pressure acting at the "centre" of each joint is necessarily vertical. On the other hand, the direction of the resultant pressure at any vertical joint of an arch ring is tangential to the curve of least resistance.

It will be further observed that the magnitude and line of action of the resultant of the forces acting on a structure, whether it be a single block or group of blocks, are independent of the *order* in which the forces are considered. Thus, it is often convenient to deal first with the vertical loads, by means of an equilibrium polygon, described so that its

angles lie on the lines of loads *produced*, as explained in paras. 87 and 88; a convenient position can thus be selected for it on the paper. By means of this polygon the line of action and magnitude of the total resultant vertical load acting on each successive joint can be ascertained. The corresponding resultant oblique loads can then be introduced in their proper places, and the line of resistance determined with fewer lines than by the method above described. This method is fully illustrated in Chap. XV., wherein the stability of walls, built to retain water, is considered.

Structures of Cemented Blockwork.

174. These structures are cantilevers, and are treated as such. The following remarks refer in the main to them. Their weight and form alone being insufficient to ensure stability as regards overturning, the tensile strength of the material must be relied on to counteract that tendency. It will be observed that the danger of sliding at a bed joint is absent in these structures, the force producing such tendency being resisted both by the force of friction at the bed joint and by the shearing strength of the material, which latter forces in every case far exceed the former. It is, therefore, only necessary to enquire into the stability of the structure as regards overturning and crushing. "*It is evident, however, that when a stress is applied to a structure of cemented blocks, sufficient to call the tensile and adhesive resistance of the material into play, the Engineer can only count upon whichever of the two resistances is the smaller.

175. "If the mortar used by the Engineer be divided into four classes, viz.:—

"(*a*). *Cement mortars*, made of one part artificial cement (such as Portland) mixed with not more than two parts sand, or of a natural cement (as Roman) with not more than one of sand;

"(*b*). *Hydraulic mortars*, one part to two of sand;

"(*c*). *Pure lime mortars*, one part to not more than three of sand.

"(All the mortar being properly mixed without excess of water and the bricks or stone well wetted), experiments and experience tend to show—

"1*st*. That cement mortars are capable, after a comparatively short time, of exerting a tensile and adhesive resistance nearly equal to the

* Wray and Seddons' "Instruction in Construction," 3rd Edition, p. 353.

Fig. 56.

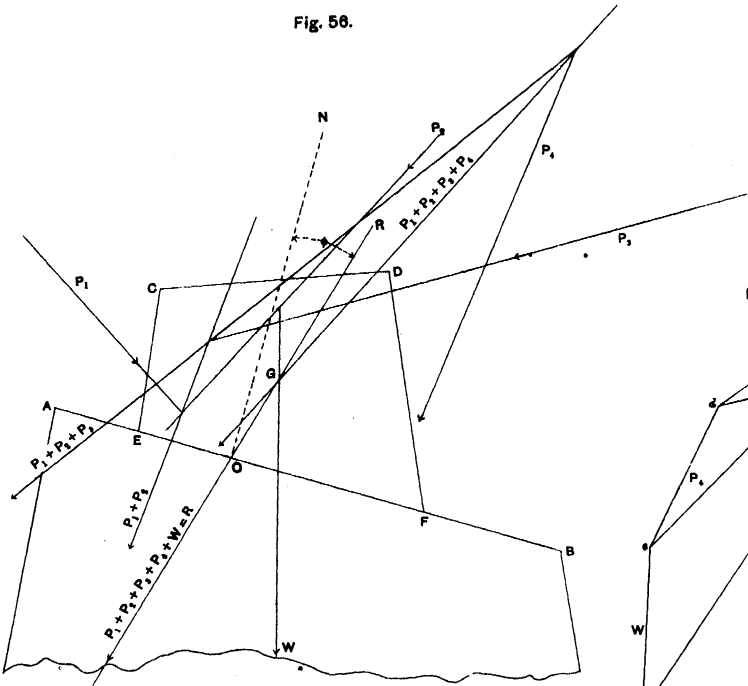

Fig. 57.

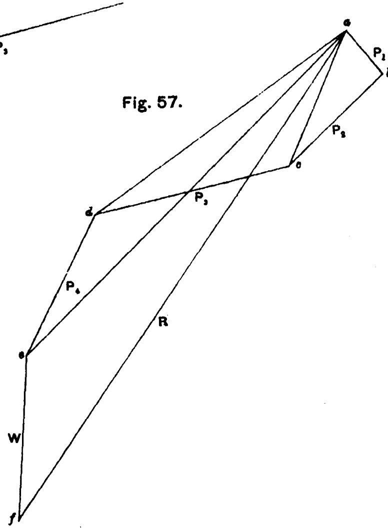

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Fig. 58.

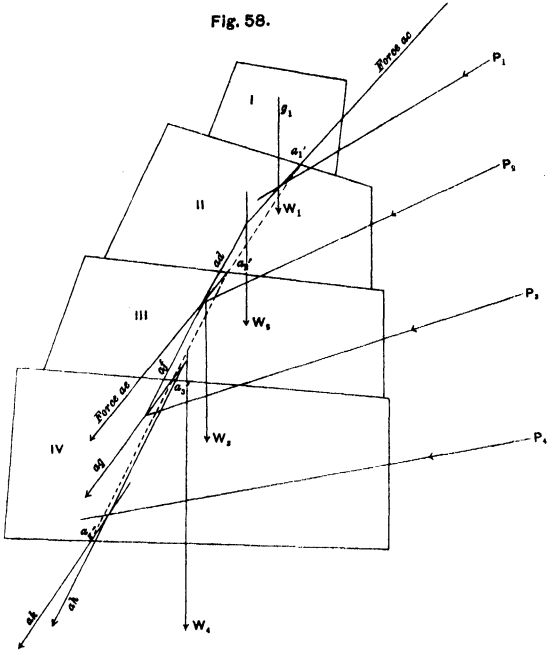
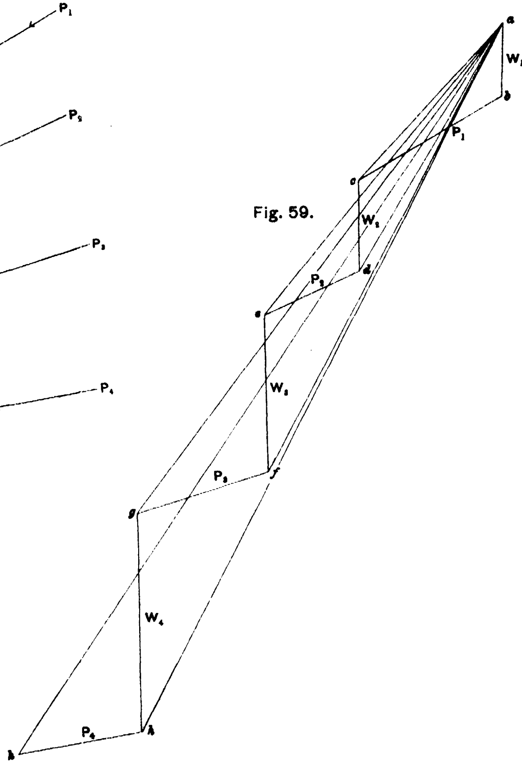


Fig. 59.



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tensile resistance of good brick, or about 280 to 300 lbs. per square inch.*

" *2nd.* That the tensile resistance which strong hydraulic mortar is capable of exerting is far in excess of the adhesive resistance of the same mortar, and that it is not safe to calculate upon the best hydraulic mortar adhering to the hardest bricks, after a period of six months, with a greater ultimate strength than 36 lbs. per square inch. The adhesion to comparatively soft bricks does not exceed 18 lbs. per square inch.†

" *3rd.* That it is not judicious in designing any structure to take into consideration the tensile or adhesive resistance of the weaker hydraulic or pure lime mortars.

176. " The *cracking and crushing resistance of brickwork* is much less than that of the bricks themselves, its value depending greatly on the nature of the mortar in which they are laid (see *Hurst's Handbook* and *Notes on Building Materials, School of Military Engineering, Chatham*). For practical purposes it may be concluded that, after allowing time for setting, say from three to six months, according to the mortar, good brickwork will begin to fail by *cracking* at a pressure of 200, 400, or 700 lbs. per square inch, according as it is laid in pure, hydraulic, or cement mortar; for *ultimate crushing*, from one-and-a-half to twice these pressures would be required."

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* Rankine's "Useful Rules and Tables," p. 195.

† Minutes of Proceedings of Institute of Civil Engineers, Vol XVII., p. 420.

CHAPTER XI.

DISTRIBUTION OF STRESS AT A PLANE JOINT.

177. If the intensity of stress acting over a plane surface be uniform and constant in magnitude, the centre of stress corresponds with the centre of figure of the surface, and the structure is subjected to simple longitudinal strain only. Conversely, if the point of action of the resultant normal pressure applied over a joint correspond with the centre of figure of the joint, the intensity of stress is uniform and constant, and if N represent the value of the resultant pressure, A the area over which it acts, and n its intensity, we have the relation $N = nA$.

On the other hand, if the centre of pressure of the cross section do not coincide with its centre of figure, but deviate from it by a certain distance, then will the distribution of the stress over the surface be unequal.

178. Now it has been universally assumed that in the latter case the distribution of stress is a *uniformly varying* one, that is, that the intensity of the stress at any point of the surface to which it is applied varies directly as its distance from some neutral line in the surface, along which there is no stress at all. Thus, if the normal stress at any point in a joint be represented by a straight line drawn at right angles to the joint's surface at that point, the locus of the extremities of the straight lines representing the stress intensities acting over the whole area will evidently lie in a plane, which will be parallel or inclined to the joint's surface, according as the neutral axis is, or is not, situated at an infinite distance from the section of the structure under consideration.

Graphical Investigation.

179. Let AB, *Fig.* 60, represent the surface cut by a plane at right angles to the plane of the joint (*i.e.*, cut by the plane of the paper), and suppose a total normal pressure N to be applied to it. Let $AB = t$, and suppose the width of the surface (perpendicular to the surface of the

cutting plane, or that of the paper) to be unity. Then N may be represented in magnitude by the area of the rectangle $AabB$, whose breadth is $AB = t$, and height $= Aa =$ mean value of $N = \frac{N}{t} = n$ (say).

180. Now the rectangle $AabB$, although it may always represent the *total magnitude* of the resultant stress acting over the area AB, can only represent the stress *distribution*, when the neutral axis K is at an infinite distance, that is, when the mean stress $\frac{N}{t}$ is constant and equal to n. The centre of stress is then at C midway between A and B; N acts along cC, that is, passes through the centre of figure of the representative rectangle $AabB$.

181. Now suppose the neutral axis K to approach AB, so that the locus ab takes up some such position as $a_1\ b_1$ (*Fig.* 61). Then the representative stress area is the trapezium Aa_1b_1B, the area of which still equals $AabB = N$, but the sides of which Aa_1 and Bb_1 represent respectively the maximum and minimum intensities of stress acting over the surface. The line of action of N, as before, passes through the centre of figure of the representative figure Aa_1b_1B, and now meets AB in some point, O_1, lying between A and C, such that AO_1 is $> \frac{t}{3}$ and $< \frac{t}{2}$.

182. In order to measure the relation between the maximum, minimum, and mean intensities of pressure, let $N =$ resultant normal pressure over the joint area, of width one inch.

t inches $=$ length of joint $=$ AB.

d inches $=$ distance of centre of resultant pressure O_1 from outer edge A of joint.

$Y =$ maximum stress intensity per square inch $= Aa_1$.

$y =$ minimum stress intensity per square inch $= Bb_1$.

Then, dividing the trapezium Aa_1b_1B into two triangles by joining the points A and b_1 and equating the moments about A, we have

$$N \times d = (n \times t) \times d = \frac{tY}{2} \times \frac{t}{3} + \frac{ty}{2} \times \frac{2t}{3}$$

$$\text{whence } y = \frac{3nd}{t} - \frac{Y}{2} \quad \ldots\ldots\ldots\ldots (1).$$

$$\text{But } n = \frac{Y + y}{2}, \text{ or } y = 2n - Y \quad \ldots\ldots\ldots\ldots (2).$$

Hence, from equations (1) and (2) we have

Maximum stress intensity $= Y = 4n - \frac{6nd}{t} = \frac{2N}{t}\left(2 - \frac{3d}{t}\right)$...(3).

Minimum ,, ,, $= y = \frac{6nd}{t} - 2n = \frac{2N}{t}\left(\frac{3d}{t} - 1\right)$ ...(4).

183. Suppose the neutral axis K to move up to, and coincide with, B, *Fig.* 62 ; then $y = 0$ and $Y = \frac{2N}{t} = 2n$; the representative figure becomes the triangle Aa_2B; and since the line of action of N passes through its centre of gravity, we shall have $d = \frac{t}{3}$. It will be noticed that in the three cases considered above, the stress intensity is all of one kind.

184. Suppose now the axis K to pass to the left of B, and fall between B and C, so that d is necessarily $< \frac{t}{3}$. Then y in equation (4) becomes negative, or the stress intensity at B becomes tensile. Since structures of uncemented blockwork are incapable of resisting a tensile strain, while those of cemented blockwork are capable of doing so, it becomes necessary to examine what will be the representative stress area in each case, in order to determine the point of action of the resultant stress and the limits of stress magnitude.

185. If AB, *Fig.* 63, represent the joint of an uncemented blockwork structure, then the triangle KBb_3 may be looked on as non-existent, and the triangle Aa_3K as the representative area of N, so that in this case $d = \frac{AK}{3}$.

As a rule, the magnitude and point of action of the resultant stress, that is, the values of N and d, are known. It is then only necessary to take $AK = 3d$ to find the position of the neutral axis K. Since the representative areas Aa_3K and $AabB$ are always equal, if $Aa_3 = Y =$ maximum stress intensity, we have $\frac{1}{2}3d \times Y = nt = N$; whence $Y = \frac{2}{3}\frac{N}{d}$.

186. It will be observed that if AB represent the joint of a structure of uncemented blockwork, equilibrium is only possible provided the resultant applied load N fall *within* the area of the joint, that is, between A and B, action and re-action being equal and opposite over the area.

187. If AB, *Fig.* 64, represent the joint of a cemented blockwork structure, then the area of the triangle Aa_3K represents the total compressive, and that of the triangle KBb_3 the total tensile, resistance that

will be called into play by the action of the applied load N. Since these resistances are opposite in kind, the difference of the triangular areas must be equivalent to the magnitude of the total resultant stress of that kind to which the larger area corresponds (in this case compressive); hence, if eK be taken equal to KB, and er be drawn parallel to Bb_3 to meet a_3b_3 in r, then will the triangular area reK = the triangular area KBb_3, and the trapezium Aa_3re represent in magnitude the total resultant compressive load N applied to the joint.

188. As a rule the magnitude and point of action of the resultant load are given, and the values of Y and y have to be determined. If, however, Y and y are given, then by setting up $Aa_3 = Y$ at A and $Bb_3 = y$ at B, perpendicular to AB, *Fig.* 65, and joining a_3 and b_3, the point K is found, and therefore the lines of action and magnitudes of the compressive resistance C, and the tensile resistance T, are known, the former acting at a distance $Ak = \frac{1}{3}$ AK from A, and the latter at $B\tau = \frac{1}{3}$ BK from B.

The position of O, the point of action of the total resultant stress N, may then be found as follows:—

(The triangles Aa_3K and KBb_3 being reduced to the same base, or to the same height, and thus represented by the length of the straight lines τc and τt, *Fig.* 65).

At τ, the point of action of T, set up τt to represent T, and τc to represent C; complete the parallelogram $\tau tt'k$, produce ct' to meet AB in O: then O is the required point of action of N. For $N = C + (-T)$, and, therefore, on the surface AB we have the system of two couples $T \times O\tau$ and $C \times Ok$ acting. Putting $Ok = x$, and $k\tau = s$, we have $T(s + x) = Cx$ or $\frac{T}{C} = \frac{x}{s+x}$.

But $\frac{T}{C}$ is, by construction, $= \frac{\tau t}{\tau c} = \frac{kt'}{\tau c} = \frac{Ok}{O\tau}$.

189. If N, or n, be eliminated from the equations of para. 182, we have $d = \frac{t}{3}\left\{\frac{Y + 2y}{Y + y}\right\}$, so that when $Y = +y$, $d = \frac{t}{2}$, as in para. 177; when either Y or y is zero, $d = \frac{t}{3}$, as in para. 183; when either Y or y is negative, d has an infinite value; and when the value of d is zero, $Y = -2y$. Since the values of T and C vary with those of Y and $-y$, it will be observed that the nearer T and C approach to equality, the further O recedes, and that when $T = C$, the points c and t,

Fig. 65, correspond, and O becomes situated at an infinite distance. The surface is then acted on by a single couple of resistance, and the case is reduced to one of pure Transverse Strain, such as is afforded by the vertical section of a horizontal beam supported at its extremities, and loaded by vertical loads. K then coincides with C, the middle point of AB.

190. It will be observed, then, that as K passes from coincidence with B to coincidence with C, O passes from distance $= \frac{1}{3}$ AB, to right of A, to an infinite distance to left of A.

When O coincides with A we have from equations (3) and (4), putting $d = 0$.

$$Y = \frac{4N}{t} = 4n = 4Aa, \text{ and } y = -\frac{2N}{t} = -2n = -2Bb,$$

$$\text{so that } \frac{AK}{KB} = \frac{Aa_4}{Bb_4} = \frac{4n}{2n} = 2 \text{ or } BK = \tfrac{1}{3} AB \; (\textit{Fig. } 66).$$

191. When O passes to left of A, d has a negative value, and equations (3) and (4) become

$$Y = 4n + \frac{6nd}{t} = \frac{2N}{t}\left(2 + \frac{3d}{t}\right) \text{ and is in compression, ...(5)}$$

$$-y = -\left(2n + \frac{6nd}{t}\right) = +\frac{2N}{t}\left(1 + \frac{3d}{t}\right) \text{ and is in tension, ...(6),}$$

or the following form may be found more convenient :—

$$Y = n\left(\frac{6d}{t} + 4\right) \text{ .... (7)}$$

$$-y = -n\left(\frac{6d}{t} + 2\right), \text{ .... (8),}$$

remembering that the elements t and d are expressed in inches, and that $n = \frac{N}{t} =$ the average intensity of normal pressure per square inch.

191a. The relations between the maximum intensity of stress Y acting at A, and minimum y acting at B, produced by a given resultant load, on a given cross section AB, and the corresponding positions of the centre of stress O, and neutral axis K, may be briefly summarised as follows :—

(1). When Y and y are equal and *both positive,* O falls at C, the centre of figure of the cross section, and K is situated at an infinite distance to the right of B.

(2). As y diminishes in value, relatively to Y, K approaches B, and O moves towards A, until, when $y = 0$, K coincides with B, and O becomes distant $\frac{1}{3}$ AB from A.

(3). As y increases numerically, *but with negative sign*, K continues to approach C and O to move still further towards A, until when, $y = -\frac{1}{2}$ Y, O coincides with A, and K is distant $\frac{1}{3}$ AB from B.

(4). As the negative value of y continues to increase, K approaches C and O moves to the left of A, until, when $y = -$ Y, K coincides with C, and O is situated at an infinite distance to the left of A. The stress-system then consists of a couple.

192. For convenience of reference, the different cases of uniformly varying-stress-distribution over a plane joint are here enumerated.

Applicable to uncemented joints only.

Case I.—Compression increasing from zero at some point *within* the joint, to a maximum at one edge, the remainder of the joint being under no stress; due to the centre of pressure being situated at a distance from the edge of maximum compression less than $\frac{1}{3}$ width of joint (para. 184).

Applicable to uncemented and cemented joints alike.

Case II.—Compression throughout joint increasing from zero at one edge to a maximum at the other; due to centre of pressure being situated $\frac{1}{3}$ width of joint from edge of maximum compression (para. 183).

Case III.—Compression throughout joint, but greater at one edge than at the other; due to the centre of pressure being situated within the centre third of the joint (para. 182).

Applicable to cemented joints only.

Case IV.—Tension on one side, and compression on the other side, of a neutral line in the joint; due to the centre of stress being situated at less than $\frac{1}{3}$ width of joint from edge of maximum stress (para. 190).

Case V.—Tension on one side, and compression on the other side, of a neutral line in the joint; due to centre of stress being situated outside the joint (para. 191).

193. It will be observed that, in order to expose the whole of a joint surface to compressive stress *only*, it is necessary to retain the centre of pressure within the centre third of the joint. This rule is always observed when designing retaining walls. It is usually observed

also when designing arch rings, though sometimes it is considered sufficient to retain the centre of pressure within the centre *half* of the arch ring.

194. In order, then, to ascertain whether a structure of given form and material is sufficiently strong to withstand the effect of a given system of applied loads, it is necessary first to determine, by methods already explained, the point of application, inclination, and magnitude of the resultant pressure acting over the surface of that joint at which crushing or sliding is most likely to occur, the component normal to the joint surface being that which causes the former, and that parallel to the surface that which causes the latter, effect. The case of sliding has been already dealt with, and may be neglected in the case of cemented blockwork. As to crushing and tensile resistance, the elements N, d, and t of equations (3), (4), (5), (6), (7) and (8) being now known, the values of Y and y can be at once determined, and compared with the corresponding values of safe crushing strength, and either safe tensile or safe adhesive strength, according as the material is weaker in tensile strength or the cement in adhesive strength.

195. The relations investigated above are applicable to joints of rectangular figure only. Professor Rankine in his Applied Mechanics, p 229, gives approximate positions of centres of pressure in joints having cross sections of figure other than rectangular. He says on p. 227—

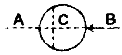
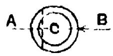
" It appears that the exact determination of the limiting position of the centre of pressure at a plane joint is, strictly speaking, a question relating to the strength of materials (*i.e.*, to their resistance to crushing). Nevertheless, an approximation to that position can be deduced from an examination of the examples which occur in practice, without having recourse to an investigation founded on the theory of the strength of materials." He then gives approximate positions for centres of pressure, under the condition *that the pressure decreases uniformly from a maximum at one edge to nothing at the opposite edge* (*Case II. above*).

Applying the above condition to joints of ordinary forms, and taking the following measurements at right angles to the neutral axis of the joint :—

t = AB = breadth of joint in direction of pressure.

d = AC = distance of centre of pressure from edge of maximum compression.

$p = \frac{1}{2}t - d =$ distance between centre of pressure and the point at which the vertical passing through the centre of gravity of the superincumbent mass cuts the bed joint.

Figures of base or joint under consideration.	Minimum value of d.	Maximum value of p	Plan.	Solid of pressure.
Square or Rectangle,	$\frac{1}{3}t$.	$\frac{1}{6}t$.		
Circle or Ellipse,	$\frac{3}{8}t$.	$\frac{1}{8}t$.		
Hollow square * (factory chimney),	$\frac{1}{6}t$.	$\frac{1}{3}t$.		
Circular ring * (factory chimney),	$\frac{1}{4}t$.	$\frac{1}{4}t$.		

The solids of pressure for the different sections graphically represent the stress decreasing uniformly from a maximum at one edge to *nil* at the other. The centre of pressure passes through the centre of gravity of such a solid, and cuts its base at a distance p from the centre of the base, or from the point in which the vertical passing through the centre of gravity of the structure above any joint cuts that joint.†

196. The simplest method of applying these principles in any particular case, is to draw on a large scale the proposed structure, and to test it for stability, altering the dimensions and form until the required degree of stability is obtained.

When the applied pressures and the bed joints of the structure are all horizontal, this can be readily done by equations, as in the Examples of Chimneys, which will be found in the next Chapter; but when either the pressures or the joints are inclined, it will generally be found simpler to employ graphic methods.

197. For structures subjected to the action of vertical loads only and for ordinary structures, or those whose lines of resistance are not very oblique to the bed joints, it is sufficient to measure the intensities of stress at these joints only. But for large reservoir walls, or import-

* These values of d and p are in practice correct enough for thin rings as in factory chimneys. Rankine's Applied Mechanics, p 229

† The above is taken from Wray and Seddons' "Instruction in Construction," 3rd Edition, p 330

ant structures whose lines of resistance are oblique, obliging the plane of the wall's face to be built with a "batter," that is, an inclination to the vertical, Professor Rankine considers that the safe limits of stress should decrease with the increase of the "batter." *See* Chapter XV., and Appendix B.

Analytical Investigation.

198. The investigation, given above, of the distribution of stress over a rectangular area may be regarded as a semi-graphical one. The general analytical treatment may be briefly stated as follows:—(Compare Chap. XIII. of Vol. I., and Rankine's Applied Mechanics, 3rd Edition, para. 91, p. 76).

If the state of stress shown in *Figs.* 61, 62 and 66, be represented slightly differently, as in *Fig.* 67, that is, by setting off a length equal to the mean stress n, immediately below a_1b_1 of *Fig.* 61, or a_2B of *Fig.* 62, or a_4b_4 of *Fig.* 66, and drawing ab parallel to a_1b_1 (or a_2B, or a_4b_4, as the case may be) so as to pass through C, then it will be seen that the state of stress, in each case, may be supposed to be made up of two parts, viz.:—

(A). A uniform stress, represented by the parallelogram a_4abb_4 of *Fig.* 67, whose mean intensity Cc is the mean intensity of the entire stress, and whose centre of stress is at C, the centre of figure of the cross section under consideration. The mean intensity

$$= n = \frac{\text{Total Stress}}{\text{Area}}.$$

(B). A uniformly varying stress, represented by the equal triangular areas aCA and CBb, which stand for stresses of opposite kinds, so that the total resultant magnitude of this second stress is zero. The intensity of this stress at any point is the deviation of the actual intensity of stress at that point from the mean intensity. Thus at A the intensity of this stress is $\mathrm{A}a = \mathrm{A}a_4 - a_4a$. The *actual* intensity of stress at any point of the surface, then, will be measured by the algebraic sum of the intensities (A) and (B). Thus, if Y and y as before represent the maximum and minimum intensities respectively, we have

$$\mathrm{Y} = \mathrm{A}a + aa_4, \quad\text{.....(1).}$$

$$y = \mathrm{B}b - bb_4, \quad\text{..... (2).}$$

The system of stress (A) is equivalent to a total resultant force N,

directly applied along the neutral surface, so as to meet the surface AB at C.

The system of stress (B) is familiar as representing the state of stress of a beam subjected to pure transverse strain (para. 198, Vol. I.), where C is a point in the neutral surface of the beam. The moment of resistance, $\mathfrak{M}$, of such a beam, if of isotropic material, we know to be measured by $\frac{f_b}{y_b}$ I, where y_b measures the distance of the extreme fibre from the neutral surface, and f_b the intensity of stress to which that fibre is exposed ($f_b = fy_b$ if f is the intensity of stress at unit distance from the neutral axis), and I represents the value of the moment of inertia of the cross section about an axis passing through its centre of figure C, a point in the neutral surface above referred to.

If, then, we suppose the neutral surface passing through C to be known and described, and resolve all the forces acting on the structure at the points where they meet that surface parallel to it and to the surface AB under consideration, then will the sum of the former components (those acting along the neutral surface) be measured by N, and the moment M of the latter will be resisted by $\mathfrak{M}$, so that we shall have $M = \mathfrak{M} = \frac{f_b}{y_b} I$, in which f_b corresponds to Aa or Bb of *Fig.* 67, so that we shall have from equations (1) and (2)

$$Y = \frac{M}{I} y_b + n, \qquad (3).$$

$$y = \frac{M}{I} y_b - n, \qquad (4).$$

If now O be the point in which the resultant R of the whole system of forces acting on the structure meets the plane of the surface AB, and if we suppose R at O to be resolved, as before, parallel to that surface and to the neutral surface above referred to passing through C, then will the former component of R be equal to N, and the moment of the latter about O be measured by M.

Then we shall have $OC = \frac{M}{N}$, since $OC \times N = M$,

or, using the symbols d and t of para. 182,

$$OC = -d + \frac{t}{2} = \frac{M}{N}.$$

$$\text{Thus, } M = N\left(\frac{t}{2} - d\right).$$

The value of I for a rectangular section, whose breadth is b, and depth d, is (para. 209, Vol. I.) $\frac{bd^3}{12}$. In the case under consideration, $b = 1$ and $d = t$, so that $I = \frac{t^3}{12}$, and f_b being equal to $\frac{t}{2}$

we have $$\frac{M}{I} y_b + n = N \left(\frac{t}{2} - d\right) \times \frac{12}{t^3} \times \frac{t}{2} + \frac{N}{t}$$

so that, $$Y = \frac{N}{t}\left(3 - \frac{6d}{t} + 1\right) = \frac{2N}{t}\left(2 - \frac{3d}{t}\right)$$

which is the same expression as equation (3), para. 182. In a similar way, equation (4) of this para. may be shown to be the same as equation (4) of para. 182. So also for equations (5) and (6) of para. 191.

199. In para. 57(b), Vol. I., the case of a "short pillar" is considered, which is so loaded longitudinally that the line of action of the total resultant load W, while parallel to the axis of the pillar, yet deviates from it by a certain distance x_0, and it is stated, but without proof, that if

x_1 = distance in inches of the point of greatest stress from the axis of the pillar

I = moment of inertia of cross section, whose area is A, relatively to its neutral axis,

then, $$\frac{\text{Mean intensity of stress}}{\text{Maximum intensity}} = \frac{1}{\left(1 + x_0 \frac{x_1 A}{I}\right)}$$

Comparing this case with what has been stated in the previous paragraph, we see that

W = total resultant load and corresponds with N, so that $n = \frac{W}{A}$.

x_0 = deviation of centre of pressure from centre of figure of cross section, and corresponds with OC of the previous paragraph, so that $x_0 = \frac{M}{W}$, and therefore $M = Wx_0$.

In this case the resultant applied load is parallel to, what in the previous paragraph is called, the neutral surface passing through C.

x_1 = distance of extreme fibre from centre of figure of cross section, and corresponds with y_b of previous paragraph.

I is the same as in previous paragraph, so that

$$\frac{M}{I} y_b \text{ becomes } x_0 x_1 \frac{W}{I}.$$

Fig. 60.

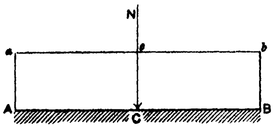


Fig. 61.

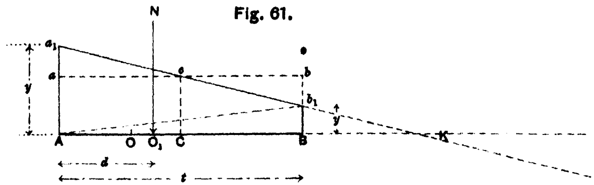


Fig. 62.

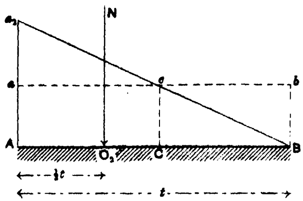


Fig. 63.

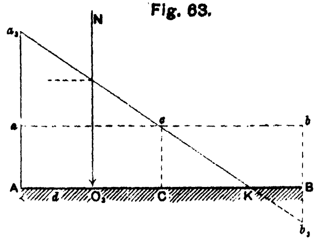


Fig. 64.

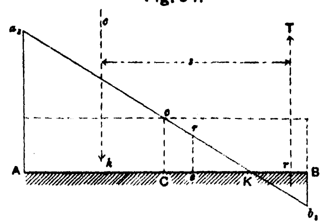


Fig. 65.

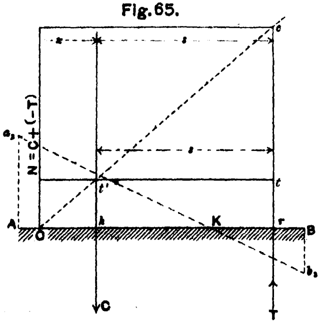


Fig. 66.

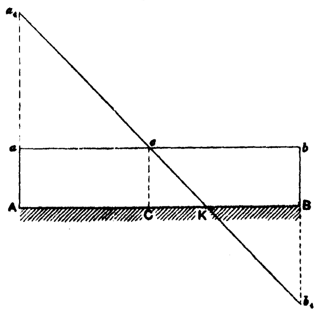
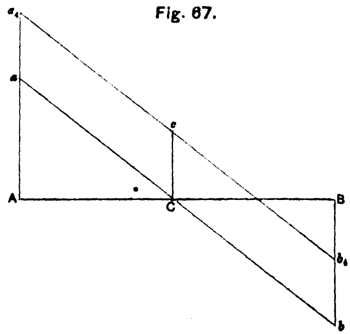


Fig. 67.



Litho. T. C. Press, Roorkee.

Hence, the maximum intensity of stress of equation (3) becomes
$Y = n + \frac{M}{I} y_b = \frac{W}{A} + x_0 x_1 \frac{W}{I}$
and, therefore, we have the ratio

$$\frac{\text{Mean intensity of stress}}{\text{Maximum intensity of stress}} = \frac{\frac{W}{A}}{\frac{W}{A} + x_0 x_1 \frac{W}{I}} = \frac{1}{1 + x_0 \frac{x_1 A}{I}}$$

which is that given in para. 57(*b*), Vol. I., above referred to.

N.B.—It is to be particularly observed that, throughout this Chapter, *the symbol* d (representing the distance of the centre of pressure from the joint's edge) *is measured from that edge of the joint towards which the centre of pressure lies, that is, the edge at which the maximum intensity of pressure occurs*, and if it be desired to adapt the equations to the case in which d is measured from the edge of minimum intensity of pressure, then $t - d$ must be substituted for d. For instance, if d be put $= \frac{2}{3} t$ in equation (3), para. 182, then $Y = O$, which is contrary to hypothesis, but if the equation be written $Y = \frac{2N}{t}\left\{2 - 3\left(\frac{t-d}{t}\right)\right\}$, and d be put $= \frac{2}{3}t$, then $Y = \frac{2N}{t}$, which is in accordance with the contents of the Chapter.

CHAPTER XII.

TALL CHIMNEYS.

200. The following remarks on Factory Chimneys are, by permission, taken from Colonels Wray and Seddons' "Instruction in Construction," 3rd Edition, p. 338, *et seq.*

Tall chimneys are regarded as structures of *uncemented* blocks, for reasons which will subsequently appear.

The only force tending to overturn a chimney or make it slide on any bed joint is the pressure of the wind. It is usual to assume that the wind blows horizontally, and with the same force at all heights above the ground, although its velocity is sensibly diminished by friction against the earth's surface.* Again, the surface exposed to the wind is regarded as being vertical, ignoring the external batter given to tall chimneys. Both of these errors are on the side of safety.

201. For the possible wind pressure to be provided for, *see* para. 116 (2), Vol. I. An ample allowance should be made, as it must be remembered that in high winds tall chimneys rock considerably, their caps often swinging through several feet, whilst the danger lies in the possibility of an exceptionally heavy gust happening to strike the shaft when in full swing to leeward.

202. The chimneys in the following Examples, both of which are in existence, can resist wind pressures from base to summit of 49·4 lbs. and 44 lbs., respectively, without the pressure at the edges of the bed joints exceeding the limit of safety, which in most cases would afford ample security, especially when it is not very exposed, or the base is more or less protected by surrounding buildings.

* Mr. E. Douglas Archibald, in *Nature*, 11th January, 1883, gives the following as the most reliable formula for deducing the velocity of the wind at any height from its recorded velocity at a given height—

$$\frac{V}{v} = \sqrt[4]{\frac{H}{h}}$$

where V and H are the velocity and height at the upper level, and v and h those at the lower level.

The great St. Rollox chimney at Glasgow, designed by Professor Rankine, is capable of resisting 55 lbs. at its weakest joint.

203. That the cylindrical does not offer nearly so much resistance to the wind as the rectangular form is well known, though the actual decrease of pressure is not known exactly.

Rankine says,* but without giving any proof, that the actual pressure against the side of a cylinder is about one-half of the total pressure against a diametral plane of that cylinder; but as any error should be on the side of safety, it is recommended on the following grounds to take three-fourths instead of only one-half.

If the surface of half a cylinder is treated as a series of 6 inclined planes, *Fig.* 68, and the horizontal force (P_h) of the wind against each inclined surface, taken from the Table, para. 116, Vol I., *the effect of the wind against the surface of a cylinder is found to be about ·74, or say* $\frac{3}{4}$, *of what it would be against a diametral plane of the cylinder, or against a square chimney with sides equal to the diameter of the round chimney.*

Against an octagonal chimney, the effect of the wind, by the same process, is found to be about ·76 of that against a square shaft of the same diameter.

Again, the normal component of a horizontal wind force of intensity p, acting on the surface of a cylinder, may be represented by $p \cos a$ (where a is the angle which the direction of p makes with the normal to the curve), hence, neglecting the component tangential to the surface of the cylinder, the portion of $p \cos a$, acting in the direction of the wind, and, tending to overturn the chimney, is $p \cos^2 a$. and the sum of these forces acting on the semi-circumference of a chimney *whose diameter is unity* $= 2 \times \frac{1}{2} \times p \int_0^{\frac{\pi}{2}} \cos^2 a \, da = \cdot 7854\ p$, or practically the same as above (radius being $\frac{1}{2}$).

204. No calculations for sliding on the bed joints are required, since even in the smallest chimneys, only half a brick thick, for every foot of flat vertical surface exposed to the wind, more than a cubic foot of brickwork, weighing about 110 lbs., would have to be moved. Now, taking the co-efficient of friction for wet mortar as low as ·5 (*Useful Rules and Tables*, p. 186, and Rankine's *Applied Mechanics*, p. 211), the resisting force would be more than $110 \times \cdot 5 = 55$ lbs.; therefore, even the smallest $4\frac{1}{2}$ inch brick chimney would resist a 55 lbs. wind pressure, and a 9 inch chimney nearly twice as much.

No calculations are required as regards overturning about the edge of any bed joint, since this could never occur if the following conditions

* Rankine's "Applied Mechanics," 3rd Edition, p. 210

with regard to the opening of the bed joints and resistance to crushing, are fulfilled.

Long before actual overturning could take place the bed joint of minimum stability would begin to open up at the windward edge, affording additional hold to the wind, besides admitting cold air and wet, and so injuring the draught of the flue; for, owing to the tendency of such structures to rock in high winds, it is never safe to reckon upon either the tensile or adhesive strength of the mortar. Hence, in designing chimneys, no tension should be allowed in any part of any bed joint.

When the centre of pressure is kept within the limits which ensure no tension being set up in any bed joint, the maximum intensity of pressure is at the leeward edge of some joint, and equals twice its mean intensity, or twice its intensity had it been uniformly distributed over that bed joint (para. 193); hence, if no bed tends to open under the wind pressure, and if the resistance the outer edges of the bed joints are capable of offering to compression is uniform throughout the structure, any undue yielding under compression, if it takes place at all, will occur at the leeward edge of the joint of maximum compression, which may, or may not, coincide with the joint of maximum stability.

205. From the above considerations, the two following rules are arrived at for the stability of chimneys:—

1. *The centre of pressure at the joint of minimum stability must remain within the limits which allow of the pressure varying from nil at the windward edge to a maximum at the leeward edge.*

2. *The maximum intensity of pressure at the leeward edge of the joint of maximum compression must not exceed the safe limit of resistance of the materials under compression.*

Bends in Flues and their Junctions with Chimneys.

206. As sharp bends in flues check the draught, they should always be curved round to the new direction as easily as possible.

207. An important point, generally neglected in factory chimneys, is the curving of the top of the horizontal flue into the vertical chimney flue, which, if properly done, would increase the upward draught, and in many cases lead to economy by enabling the height of the shaft to be reduced.

208. When two flues enter the base of the shaft from opposite direc-

tions, a *mid feather*, generally consisting of a vertical firebrick dwarf wall, is carried across the shaft at least to the top of the flue openings, to keep the two blasts from interfering with each other; this would not be required if the flues were properly eased into the shaft.

209. In bringing a flue through the wall of a factory chimney, care must be taken not to reduce the bearing area of the shaft below the limit of safety; this can be met by adopting a stronger section for the base of the shaft where pierced by the flues, or, at any rate, by thickening the cheeks of the flue opening.

EXAMPLE I.

Circular Brick Chimney.

210. It is desired to inquire into the safety of a circular brick chimney of the dimensions shown in *Fig.* 69 standing on a bed of Portland cement concrete, by determining—

1st. The maximum intensity of pressure on the brickwork at the leeward edge of the joint of maximum compression, when the pressure at the windward edge is reduced to *nil.*

2nd. The maximum intensity of pressure on the foundations under the same conditions.

3rd. What wind force would reduce the intensity of pressure at the windward edge of the bed joint of maximum compression to *nil.*

Remarks.

In this case it is evident that, if there is no tendency for the joint to open at the base, there will be no tendency for any joint to open.

The weight of the brickwork will be taken at 112 lbs. per cubic foot.

The independent firebrick lining only affects the weight on the foundations.

It is evident from *Fig.* 69 that the mass above every successive bed joint, from the top downwards, increases more rapidly than its bearing area; and hence, that the maximum compression on the brickwork will be at the base of this shaft, *i.e.*, at 136 feet from the top.

With a circular chimney shaft the centre of pressure at any bed joint at which the pressure varies from *nil* at one edge to a maximum at the other, is at $\frac{1}{4}$ the diameter of the shaft from the edge of maximum compression, *see* para. 195.

Calculations.

(1). *Maximum Intensity of Pressure at Base.*

Cubic feet of brickwork $= 136' \times \frac{\pi}{4}(11{\cdot}5^2 - 7{\cdot}3^2) = 8{,}378$ cub. ft.

Weight of shaft $= 8{,}378$ cubic feet $\times$ 112 lbs. $= 938{,}336$ lbs.

Area of base of shaft $= \frac{\pi}{4}(15^2 - 9^2) = 113$ ft. super.

Mean pressure on base $= \frac{938{,}336}{113} = 8{,}304$ lbs. per ft. super.

Maximum pressure at windward edge on base, that at the leeward edge being *nil*, $= 2 \times$ mean pressure $= 16{,}608$ lbs., or 7·4 tons per foot super., or 115 lbs. per inch super.

This would give a factor of safety of a little over 3 for good stock brickwork in hydraulic lime mortar, against cracking, and much more against crushing, para. 176; whilst the Portland cement foundations would, if properly made, be even stronger than the brickwork.

(2). *Maximum Intensity of Pressure on Foundations.*

When the centre of pressure is at $\frac{1}{4}$ the diameter, or $3\frac{3}{4}$ feet from the edge of the base, the resultant of the wind pressure and the weight of the shaft, cuts the base of the concrete foundation at nearly $\frac{1}{3}$ of its width from one edge, thus producing a maximum compression at that edge of about twice the mean pressure.

Mean pressure at base of concrete due to shaft $= \frac{938{,}336 \text{ lbs.}}{25' \times 25'} = 1{,}501$ lbs.

Maximum ditto $= 2 \times 1{,}501 = 3{,}002$ lbs. per foot super.

Adding the weight of the concrete, at 120 lbs. per cub. ft., which for a thickness of 5 feet gives 600 lbs. per foot super.; the weight of the firebrick inner shaft, at 112 lbs. per cub ft., which works out to 84 lbs. per foot super. on the foundations below the concrete; and, say $\frac{3}{4}$ of the weight of the 10 feet of earth resting on the edge of the concrete (since it is partly supported by the surrounding earth), which taken at 120 lbs. per cub. ft., adds 324 lbs. to the pressure per foot super. on the foundations; we have

Maximum pressure on foundations $= 3{,}002 + 600 + 84 + 324 = 4{,}010$ lbs., or nearly 1·8 tons per foot super., which, for the limit of pressure at one edge, is not too great, on a good sound foundation, such as firm gravel or sound hard clay.

(3). *Wind force required to reduce the pressure at the windward edge of the base of the shaft to nil.*

The effect of the wind upon a circular surface is taken as $\frac{3}{4}$ of its effect upon a flat surface (para. 203).

The wind force concentrated at its centre of pressure, *i.e.*, at the centre of gravity of the portion of the shaft above ground, acts with a leverage of 67 feet above the base.

Equating the moment of wind pressure about the base of the shaft, with the moment of the weight of the shaft itself about the point at which the centre of pressure cuts the base, which, under the condition that the pressure at one edge is reduced to *nil*, is at $\frac{1}{4}$ the diameter, or at $3\frac{3}{4}$ feet from the centre of the shaft, we have

$$\tfrac{3}{4} \times \text{wind pressure} \times 126' \times 11{\cdot}25' \times 67' = 938{,}336 \text{ lbs.} \times 3{\cdot}75 \text{ ft.}$$

$$\therefore \text{wind pressure} = 49{\cdot}4 \text{ lbs.}$$

Conclusions.

As the calculated maximum intensities of pressure are within safe limits, and can only be attained under a wind force of nearly 50 lbs. per square foot, acting at the same moment over the entire height of the shaft above ground, it is evident that the stability of the structure is insured, without calling upon the mortar to exercise any tensile or cohesive strength whatever.

An additional element of safety is due to the vertical component of the wind striking on the slightly battered surface of the chimney, which for facility of calculation has been treated as vertical.

EXAMPLE II.

SQUARE BRICK CHIMNEY.

211. A square brick chimney has a side elevation and central vertical section (omitting projections of cap, and without the fire-brick lining), as shown in *Figs.* 70 and 71, the batter being half an inch to a foot.

Inquire into its stability by determining—

1*st.* What wind pressure the chimney is capable of bearing when the position of the centre of pressure, at the joint of minimum stability, is at the limit consistent with there being no tendency of the joint to open.

2nd. Whether the maximum intensity of pressure, at the leeward edge of the joint of maximum compression, is limited to what is necessary for safe resistance to crushing.

The axis of the chimney is assumed to be vertical, and the strength of the mortar is neglected.

Remarks.

As regards resistance to overturning, or rather to the opening of the bed joints, it is evident from the small increase in the thickness of the brickwork at the bed joints AB, CD, EF, that there will be less stability at GH than at any joint above GH; whilst below GH the stability of the shaft is greatly increased by the projecting buttresses; so that it will only be necessary to determine the wind pressure required to force the centre of pressure to its extreme limit, at the joint GH.

As regards failure by crushing, it is only necessary to inquire into the maximum intensity of pressure at the same joint, GH, as below it the bearing area increases considerably, whilst the centre of pressure does not approach so near to the outer edges of the joints.

Preliminaries.

Let p = intensity of horizontal wind pressure.
„ H = height of shaft exposed to wind above joint of minimum stability.
„ h = height of centre of pressure of wind above same joint.
„ D = external diameter of shaft at same joint.
„ d = external diameter of shaft at mean height above same joint.
„ W = weight of portion of shaft under consideration.
„ m = leverage of W about centre of pressure when the latter has reached its limit

Resistance to Bed Joints opening.

The shaft being square, in order that there may not be any tendency for the joints to open, the centre of pressure must never approach nearer the edge of the bed joint of minimum stability than $\frac{1}{6}$D (para. 195).

Taking this limit of stability, and equating the Moment of wind pressure with the Moment of the weight of shaft,

$$pdH \times h = W \times m, \text{ ....(1).}$$

GH being the bed joint of minimum stability—

At GH, $m = (\frac{1}{2} - \frac{1}{6})\, D = \frac{D}{3} = \frac{11}{3}$ feet.

Above GH, height of shaft $= H = 90$ feet.

" " mean outer diameter $= d = 9{\cdot}1$ feet.

" " the height h of the centre of wind pressure or centre of figure KGHL (*Fig.* 71) $= 41{\cdot}8$ feet.

" " $W = 422{,}946$ lbs., obtained as follows :—

At mean height above EF, width from out to out $= 8{\cdot}7$ feet.

" " " " in to in $= 5{\cdot}7$ feet.

" " " area of brickwork, deducting recesses,

$$= 8{\cdot}7^2 - 5{\cdot}7^2 - 4 \times 4{\cdot}5' \times \tfrac{3}{8}' = 36{\cdot}45 \text{ sq. ft.}$$

Weight of shaft above EF $= 36{\cdot}45' \times 72' \times 110$ lbs. $= 288{,}684$ lbs.

Between EF and GH, height of shaft $= 18$ feet.

" " mean width, out and out $= 10{\cdot}6$ feet.

" " " in and in $= 6{\cdot}1$ feet.

" " mean area of brickwork deducting recess,

$$= 10{\cdot}6^2 - 6{\cdot}1^2 - 4 \times 5{\cdot}5' \times \tfrac{3}{8}' = 66{\cdot}9 \text{ sq. ft.}$$

Weight of shaft between EF and GH $= 66{\cdot}9' \times 18' \times 110$ lbs. $= 134{,}262$ lbs.

Total weight above GH $= 288{,}684 + 134{,}262 = 422{,}946$ lbs.

By substituting in Equation (1)—

$$p \times 9{\cdot}1 \times 90 \times 41{\cdot}8 = 422{,}946 \times \frac{11}{3}$$

$$\text{whence } p = 45{\cdot}3 \text{ lbs. per sq. ft.}$$

By proceeding in the same way at the joint EF, it would be found that p would exceed 45·3 lbs., showing that the joint of minimum stability is at GH.

Resistance to crushing.

At the bed joint GH, where the intensity of pressure will be a maximum—

Normal pressure $=$ weight of shaft above GH $= 422{,}946$ lbs.

Area of bearing surface, deducting recesses, $= 11^2 - 6{\cdot}5^2 - 4 \times 6{\cdot}5' \times \frac{3}{8}' = 68$ sq. ft.

Mean intensity of pressure per square inch $= \dfrac{422{,}946}{68 \times 144} = 43{\cdot}2$ lbs. per square inch, which would give a maximum intensity of 86·4 lbs. per square inch, leaving an ample factor of safety, seeing that stock brickwork in lime mortar is capable of bearing 400 lbs. per square inch before beginning to yield (para. 176).

The chimney is therefore perfectly safe against crushing under the wind pressure that brings it to its limit of stability as regards the tendency of the joints to open.

Summary.

It appears that this chimney, supposing no reliance to be placed on the tenacity and adhesion of the mortar, is safe against a wind pressure of 45 lbs. per square foot of flat surface directly opposed to it.

If the wind pressure were to exceed 45 lbs. per square foot, the centre of pressure would approach somewhat nearer to the leeward side of the chimney, the maximum intensity of the pressure would be increased, and tension would be set up on the windward side.

To fulfil the conditions of stability under 50 or 56 lbs. wind pressure over its entire height above GH, the dimensions of the shaft should be slightly increased.

Fig. 69.

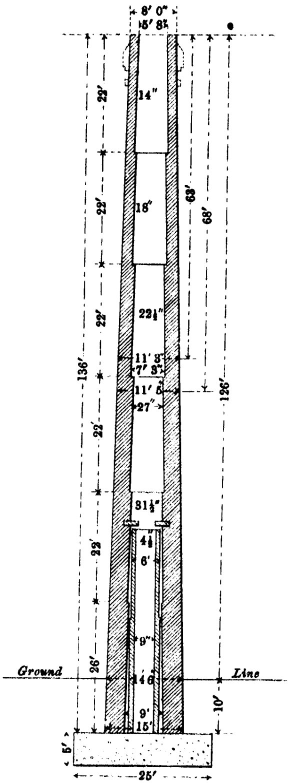


Fig. 68.

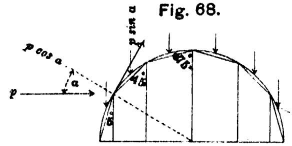


Fig. 70.

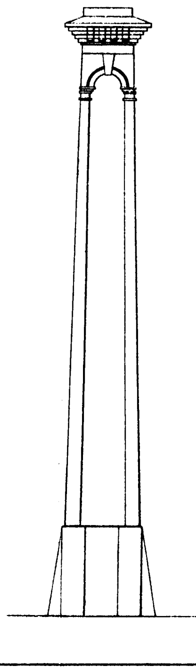
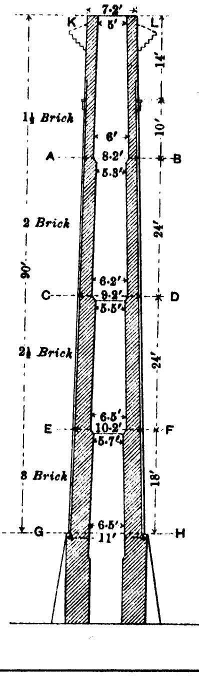


Fig. 71.



CHAPTER XIII.

EXAMPLES OF BLOCKWORK STRUCTURES.*

EXAMPLE I.

BUTTRESS.

212. A buttress of strong brickwork, weighing 112 lbs. per cubic foot, of the form and dimensions given in *Fig.* 72, and of a uniform width of 5 feet, has to sustain two inclined thrusts in the positions, and having the directions and magnitudes shown in the same figure; the tenacity and adhesion of the mortar to be neglected.

Inquire into its stability by determining—

1*st.* Whether the line of resistance intersects the bed joints of minimum stability, AB, CD, and sufficiently within their outer edges to insure safety against crushing.

2*nd.* Whether the buttress is secure against sliding on the joints GH and EF.

Remarks.

The pressures P and P_1 are the thrusts due to a roof and an arch.

As regards failure by overturning, the buttress is evidently not liable to overturn at any point above K, the pressure P having no leverage above that point.

Further, by drawing a diagram and describing the curve of resistance as explained in para. 173, it will be seen that, for a structure of uniform width to which a single lateral force, as P, is applied, the curve of resistance continually approaches the outer face of the structure as it descends; hence, the buttress is more likely to overturn about the bed joint AB than at any joint above it.

Owing to the increase of width at EF, it is not so liable to overturn

* The data of these examples are taken from Wray and Seddons' "Instruction in Construction," 3rd Edition, p. 335, *et seq*

"The following enclosure walls have been actually constructed. A long length, 550 feet, of one of them (Wall No. 1), was blown over, as if it had been hinged at the footings.

"They are given as typical examples of enclosure walls, and are intended to show that, under a horizontal uniformly distributed wind force, normal to the wall, of 55 lbs. on the square foot, these walls can only remain standing by virtue of the tenacity and adhesion of the mortar.

"*Wall No.* 1, *(Fig.* 74*)*.—Weight of material = 105 lbs. per cubic foot. * * This wall was built in pure lime mortar, 1 lime to 2 sand, and was blown down about 18 months after it was built.

"*Wall No.* 2, *(Fig.* 75*)*.—Weight of material = 112 lbs. per cubic foot. * * This wall was built in grey chalk lime mortar, 1 lime to 2 sand. In this case, the wall being very low, the friction of the wind against the ground would act more than in the other cases.

"*Wall No.* 3, *(Fig.* 76*)*.—Weight of material = 112 lbs. per cubic foot. Pressure per square foot required to overturn the wall = 21 lbs. This wall was built in grey chalk lime mortar, 1 lime to 2 sand.

"*Wall No.* 4, *(Figs.* 78 to 80*)*.—Weight of material = 112 lbs. per cubic foot. Pressure per square foot required to overturn the wall = 25·4 lbs. This wall was built with brick in grey chalk lime mortar, 1 lime to 2 sand.

"From the above it appears advisable, in constructing an enclosure wall of an ordinary section, on any very exposed site, to provide a mortar of considerable tenacity and adhesion."

Examples.

The data of the following Examples are taken from Cols. Wray and Seddons' "Instruction in Construction," 3rd Edition, p. 350, *et seq.*

"*Wall No.* 5.—A long enclosure wall, 16 feet high, *Fig.* 77, and three bricks thick laid in hydraulic mortar, weight only 100 lbs. per cubic foot, is built in an exposed position where it might be called on to resist a wind pressure of 55 lbs. per square foot.

"Determine the compression per square inch on the lee side of the bed joint at the footings, and the tension per square inch on the windward side of the same joint, when the wind is blowing perpendicularly to the face of the wall, with its maximum force, and uniformly all over it.

"This wall is of an ordinary height as enclosure wall of a prison, dockyard, &c. Such walls are, however, seldom made as thick as three bricks. A uniform width has been assumed for the sake of simplicity in illustrating the principles involved.

"It may be regarded as certain that, in any but very exposed positions, the wind never acts with its maximum registered force on a long length of low wall, and even in such positions there must be some reduction in its force near the lower part of the wall, owing to the friction of the wind along the ground; however, in very exposed sites it is better to provide for the maximum registered wind pressure.

"As the wind seldom blows with its maximum registered force, and still less often with its maximum force normal to the face of the wall, it is justifiable to calculate upon the mortar having time to acquire a considerable proportion of its ultimate tenacity and adhesion to the blocks, before the wall is subjected to the greatest pressure it can ever have to bear.

"Referring to *Useful Rules and Tables*, p. 204, it would be justifiable to calculate in this case upon the *tenacity* of good hydraulic mortar one year after mixture, given at 140 lbs. per square inch, allowing a proper factor of safety (say 3, as the force is only occasional), and upon the *adhesive resistance* of the mortar being 36 lbs. per square inch with the same factor of safety.

"The ultimate crushing resistance of good hydraulic mortar is certainly not less than 500 lbs. per square inch.

Preliminaries.

"Let W lbs. $=$ weight of one foot run of wall $= 16' \times 1' \times 2\frac{1}{4}' \times 100 = 3{,}600$ lbs.

"The total normal pressure per foot run of wall at ground line $=$ W $= 3{,}600$ lbs.

"The intensity of pressure, W, being uniformly distributed, $= \dfrac{3600}{2{\cdot}25' \times 144}$ $= 11{\cdot}11$ lbs. per square inch $=$ N."

Total wind pressure acting over one foot run of wall $= 16 \times 1 \times 55 = 880$ lbs. $=$ P.

If a diagram be drawn to a large scale, d will be found to measure about 10 inches. This may be corrected by calculation as follows:—If h be height of wall in inches, remembering that W acts through the

centre of gravity of the wall and P at $\frac{h}{2}$ above ground level, also that the cross section of wall at that level is the weakest with regard to overturning, we have the relation $d + \frac{t}{2} : \frac{h}{2} : : P : W$, or, $d + \frac{2 \cdot 25 \times 12}{2} : 8 \times 12 : : 880 : 3{,}600$, whence $d = 9 \cdot 96$ inches.

Substituting these values of n, d, and t in equations (7) and (8) of para. 191, we have $\frac{6d}{t} = \frac{6 \times 9 \cdot 96}{2 \cdot 25 \times 12} = \frac{166}{75} = 2 \cdot 21$.

Hence $Y = 11 \cdot 11 \times 2 \cdot 21 = 69$ lbs. per square inch compression.

$- y = Y - 2n = 69 - 22 \cdot 22 = 46 \cdot 78$ lbs. per square inch tension.

There is therefore a factor of safety of $\frac{500}{69} = 7 \cdot 2$ against crushing, but the wall must fail by the opening of the joints on the windward side, if the wind ever blows with its maximum force normal to its face, the adhesive resistance which the materials are capable of exerting being only 36 lbs. per square inch, against a tensile stress of 47 lbs. nearly.

This source of danger might be entirely avoided by using cement mortar in the lower part of the wall.

" *Wall No.* 6.—A long enclosure wall, 15·5 feet high, and of the plan and section given in *Figs.* 78 to 80, of brick in hydraulic mortar, weighing 108 lbs. per cubic foot, is built in an exposed situation in a country where the force of the wind has been registered at 55 lbs. on the square foot.

" Determine the compression per square inch on the lee side of the bed joint at the ground level, and the tension per square inch on the windward side of the same joint, when the wind is blowing perpendicularly to the face of the wall, with its maximum force, and uniformly all over it.

" The same remarks as to wind pressure and strength of materials apply as in last Example.

" As it makes no difference, and is more convenient to deal with, the plan of the wall will be taken as in *Fig.* 80 in place of *Fig.* 79.

Preliminaries.

" As the wall is not uniform in section throughout, it is best to take as a *unit of length*, the portion 19·5 feet long, between the centres of two alternate piers.

" A height of 16 feet is taken because the coping being inclined does not offer the same resistance to the wind as a vertical surface.

"Then $W = (19 \cdot 5 \times 16 \times 1 \cdot 5 + 6 \times 16 \times \frac{3}{8}) \times 108 = 54{,}432$ lbs.
$=$ total normal pressure *per unit of length* at ground level."

Fig. 74.

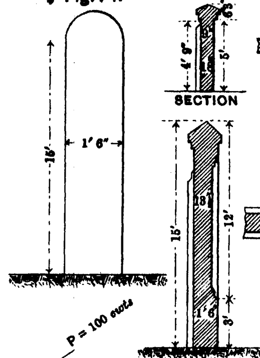


Fig. 75.

Fig. 72.

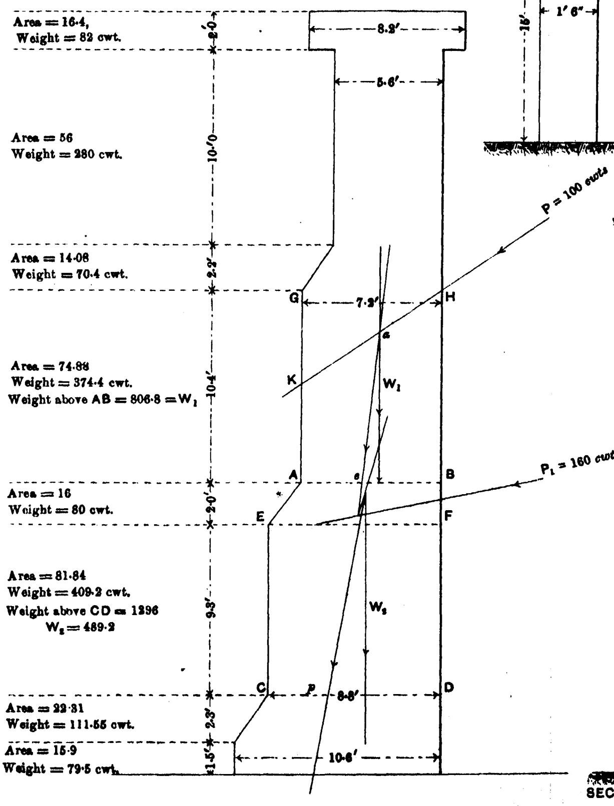


Fig. 76.

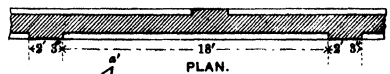


Fig. 73.

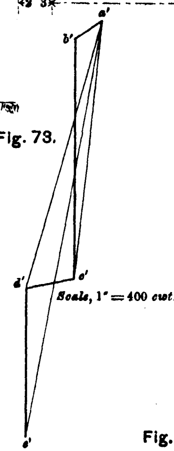


Fig. 77.

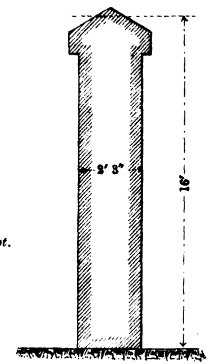


Fig. 78.

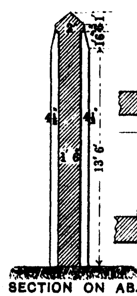


Fig. 79.

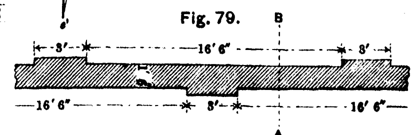


Fig. 80.

Litho. T. C. Press, Roorkee.

The area over which this is uniformly distributed

$$= (19{\cdot}5 \times 1{\cdot}5 + 2 \times 3 \times \tfrac{3}{8}) = 31{\cdot}5 \text{ sq. feet.}$$

Hence, intensity of pressure per square inch $= \dfrac{54432}{31{\cdot}5 \times 144} = 12 \text{ lbs.} = n.$

The total wind pressure *per unit length* $= 19{\cdot}5 \times 16 \times 55 =$ 17,160 lbs.

Drawing a diagram and proceeding as explained for the last Example, we have $d + 9 : 8 \times 12 :: 17160 : 54432$.

Whence $d = 30{\cdot}26 - 9 = 21{\cdot}26$ inches and is negative.

Substituting these values of n, d, and t in equations (7) and (8), para. 191, we have $6\,\dfrac{d}{t} = 7{\cdot}09$.

Whence $Y = 12 \times 11{\cdot}09 = 133$ lbs. pressure per square inch.

$- y = Y - 2n = 133 - 24 = 109$ lbs. per square inch tension.

Hence, at the leeward edge there is an intensity of pressure of 133 lbs. per square inch, and at the windward side a tensile stress of 109 lbs. per square inch.

"There is, therefore, a factor of safety of more than 3 in compression, but this wall must fail by the opening of the joints on the windward side, if the wind ever blows with its maximum force normal to the face of the wall, the adhesive resistance of the materials being only 36 lbs. per square inch to resist a tensile stress of 109 lbs.

"This source of danger may be avoided by using cement mortar in the lower parts of the wall, which, with a tensile strength of about 280 lbs. per sq. inch (*vide* para. 176), would give a factor of safety of over 2."

CHAPTER XIV.

CRANES.

214. Cranes are cantilevers, and may be regarded as structures intermediate between those employed in spanning an interval and those not so employed. They are in fact equivalent to beams or arches cut short, loaded at the free extremity and firmly fixed at the other extremity.

Figs. 81 to 83 illustrate the analogy existing between the supported beam and cantilever.

Beam A is supposed to be supported at both extremities and loaded at the middle C.

Beam B is supposed to be loaded at both extremities and supported at the middle C.

Each portion $A'C$, $A''C$ of Beam B, forms a cantilever, supposed to be fixed at extremity C, and loaded with R' at A' and R'' at A''.

The method of examining the stability of all these structures is the same in principle as that already described. A skeleton system of imaginary resultant forces, which produce an effect exactly similar to that of the system actually applied, is supposed to replace the latter, and the resulting effect on the several pieces and on the material of the structure is then examined. The joints of structures composed of open frame work are considered as belonging to the first class, those of structures of solid wood or iron as belonging to the second. In the former, the centre of resistance of a joint must correspond with the point of intersection of the axes of the several bars which meet at that joint; in the latter, the joint itself affords a certain area within which the centre of resistance may lie. Having no equivalent closing piece, such as the imaginary tension bar afforded by the abutments of an arch, or the imaginary compression bar afforded by the piers of a suspension bridge (para 19), the forces resisting the action of the bending moment must be supplied by the material of the actual pieces of the structure itself, and hence the state of strain of a cantilever becomes

reversed as compared with that of an equal portion of a similar beam or arch similarly loaded (*Figs.* 81 to 83), the upper fibres or pieces being thrown into tensile and the lower into compressive strain. Thus it will be seen that the section of maximum bending moment occurs at the point of support or fixation (para. 91), and that the curve is convex with regard to the axis of abscissæ, and not concave, as in the case of the beam or arch. This is sufficiently evident from *Figs.* 6 and 7, Ex. 1 and 2, para. 182, Part II. of Vol. I., and the accompanying *Figs.* 81 to 83.

215. Thus, if we wish to design a cantilever of uniform strength with flanges of equal width above and below, and a web connecting them, since the bending moment is, as in the case of the supported beam, resisted by the flanges, and the shearing force by the web, the form of the web, that is, the depth of successive sections of the cantilever measured parallel to the total resultant load, will, as in the case of the beam, be determined by the curve of bending moments or equilibrium polygon. If, on the other hand, the flanges are to be parallel to one another, then the same curve determines their width or thickness, as the case may be, due allowance being made for the difference of strength of the material in compression and in tension (para. 61).

The general subject of Cantilevers has been sufficiently dealt with in Part II., Vol. I.; the special application in the form of Cranes will now be considered.

216. The Jib Crane consists of a vertical standard AB, *Fig.* 84, (which represents a skeleton crane), freely moveable about a pivot at B, which is firmly imbedded in the ground. The jib BC is attached to the standard at B, just above the ground level, and usually in such a manner as to be capable of being raised or lowered by altering the length of the tension piece AC, which is sometimes made of rod-iron and sometimes of chain. BC thus becomes moveable both in a horizontal and vertical plane, and the point C (from which the load is suspended) may thus be brought immediately over the position of the weight to be raised. At C there is a block through which a chain passes, one end of which is fastened to the weight and the other carried to a winch placed somewhere near B.

217. The Derrick Crane, *Fig.* 85, is similar in principle to the Jib Crane. Its standard, AB, however, is not deeply imbedded in the ground, but guyed back by chains or rods AE_1, AE_2, which are known

as *back stays*, and may be either anchored into the ground, or more usually fastened to the two horizontal pieces BE_1, BE_2, which make a right angle (E_1BE_2) at B. The piece AC is always a chain, and by it the jib BC is raised or lowered, by means of a winch, placed near B.

218. In each of these Cranes it is obvious that, on applying the load at C, the piece AC is thrown into tensile, and the piece BC into compressive, strain, and that the upright piece AB tends to be broken across, the fibres towards the jib being compressed, and those on the opposite side extended. The standard AB becomes, in fact, a cantilever subjected, in addition to transverse strain, to the action of a resultant longitudinal shearing stress, equal in magnitude to the load applied at C. The frame ACB is likewise a cantilever, obliquely loaded at C.

219. Comparing the figure of either of these cranes with the diagram, *Fig.* 6, para. 182, Part II., Vol. I., it will be seen that it is an oblique projection of the curve of bending moments of a cantilever loaded with a single weight at its extremity; and seeing that the weight of such a crane is small compared to the loads it is called on to lift, there is good reason for so shaping it. The weight of these structures is in fact usually not taken into consideration.

220. If we take AB (*Fig.* 81) on any scale of moments to represent the moment of the load at section AB (ABC being drawn parallel to the pieces of the frame), then will xy represent the moment at any section xy.

221. Moreover, the shearing force is in this case constant, and equal to the load at C, so that if AB be taken on a scale of loads to represent it, and the parallelogram AD be drawn, then will it be a graphic representation of the shearing force acting at all sections taken parallel to the direction of W.

222. Or the case may be treated otherwise as follows:—

Taking AB to represent the greatest Load W that is ever likely to be lifted by the crane, and drawing AC, BC parallel respectively to the stay and jib, we have the sides of the triangle BAC parallel to the pieces of the frame and also to the forces acting at the point C, so that CA will represent the tension in the stay and BC the compression in the jib (both of which are constant if the weight of the structure be neglected), on the same scale that AB represents the load at C. As ABC also represents the figure of the frame, consider the state of equilib-

rium at any section xy. Through y draw bd, and through B draw BD (meeting xy in x'), both parallel to AC. Then the bending moment at section xy is represented by the area of the parallelogram xD, and is resisted by the tension in CA and the compression in BC. Resolving the latter stress parallel to AC and to the direction of W, we see it is equivalent to forces BD, DC, that is, to a force equal and opposite to CA acting at y, together with a shearing force DC, equal and opposite to AB, and acting along the section xy.

Hence the resisting couple is obviously represented by the parallelogram Ad. But taking the part xd away, being common to the two parallelograms Ad, xD, we have the remaining area Ay equal to the remaining area yD, which obviously is the case by the properties of the parallelogram. Hence, the bending moment, which is represented by the parallelogram xD, is always equal to the moment of resistance represented by Ad.

223. If we wish to measure the state of stress of the upright cantilever AB, we join AD, *Fig.* 84. Then the triangle ABD is a graphic representation of the curve of bending moments acting on the standard AB (ordinates being measured parallel to BD, which should represent the maximum bending moment at B). Thus the bending moment at any section, as b, whose plane is taken parallel to BD, is measured by bd' on the same scale of moments as BD measures the bending moment at B.

There is also the shearing force due to the load at C and equal to it acting along AB.

224. It should be observed that the above measurements suppose the pulley supporting the load at C to be frictionless. Practically this is not the case. The chain passes from the pulley at C to a winch or drum at E, and the result of friction in the pulley is to make the stress in EC always greater than the weight (W) when it is being lifted, and less when it is being lowered; and if the pulley be not in good working order and not properly lubricated, this difference may be considerable. The effect upon the pieces of the crane is to diminish the stress in the stay and increase that in the jib.*

225. The Wharf Crane, *Fig.* 87, is much used in docks and quays for loading and unloading ships. Its foot rests in a pivot at G, wherein it can revolve in a horizontal plane around a vertical axis. The portion of

* "The Design of Structures," Anglin, p. 292.

the crane above ground forms a curved cantilever and that below a straight one. These cranes are usually made of wrought-iron or steel, their cross section being either rectangular, circular, or of H form. Their sides are often open and braced, *Fig.* 88.

The lifting chain passes over a pulley at C and is led over a series of pulleys, or rollers, placed at intervals along the top and back of the jib (*i.e.*, the side opposite to that on which the weight hangs).

The portion, or pieces, of the structure on the side removed from the load will be in tension and those towards the load in compression.

226. Thus considering any section xy, *Fig.* 87, whose plane is parallel to the direction in which the load acts, *i.e.*, vertical, and distant x from it, we have an active bending moment $= Wx$, which must be resisted by the resultant tensile stress in the upper fibres, and the resultant compressive stress in the lower fibres, of the structure.

Example of Braced Iron Crane.

227. *Fig.* 88 represents the frame diagram of a braced crane, composed of open frame work, and *Fig.* 89 shows the corresponding stress diagram on a scale of 20 tons to the inch, the weight W suspended at c being supposed to weigh 20 tons. The stresses developed in the several pieces of the frame are shown in the accompanying Table.

A few words are necessary in explanation of the stress diagram, *Fig.* 89.

Bow's method of lettering has been employed (para. 258a, Vol. I.)

It will be noticed that the structure is supported vertically at the point x, horizontally at the ground level y, and that the vertical load W ($=$ 20 tons) is hung at c (*Fig.* 88).

The vertical load W, hung at c, must then, if equilibrium obtain, be counterbalanced by an equal and opposite vertical resistance or shearing force, W, acting at x. The suspension of W at c causes a tendency of the frame to revolve round x *towards* c, and consequently produces a horizontal pressure acting through y. This active horizontal force, which we may call H, must be balanced by an equal and opposite passive force H, acting through x, if the frame is to remain at rest, otherwise the couple W . xy, which tends to produce revolution in the direction of the hands of a watch, will not be resisted by an equal and opposite couple, such as H . xy, tending to produce revolution in a contrary direction.

The structure may, then, be regarded as being kept in equilibrium by this pair of equal and opposite couples.

But the two passive forces H and W acting at x are equivalent to a single resultant R, say, which resultant must be equal and opposite to that of the two active forces W and H acting at z.

The structure then may equally well be regarded as being kept in equilibrium by the equal and opposite forces R, the one active, in direction zx, and tending to cause the frame to revolve about its centre of mass in the direction in which the hands of a watch move, the other passive, acting in direction xz, and tending to produce revolution in an opposite direction.

The triangle xyz, moreover, evidently has its sides proportional to the magnitudes of the forces W (yx), H (yz), and R (zx).

The triangle XYZ of *Fig.* 89 is similar to the triangle xyz of *Fig.* 88, and has its side XY = 1 inch, to represent 20 tons. XY then represents the vertical load applied at c, and YZ the horizontal pressure applied at b.

We see, then, that while the upper and left sides of the frame suffer no change in external loading, a change occurs on the right side at b. Therefore the external area beyond the upper and left sides may be denoted by (X) throughout, and that of the area to the right of the lower and right sides of the frame—the part above the horizontal ab by (Y), that below it by (Z), *vide Fig.* 88.

The effect at point b of the vertical load W, hung at c, may be measured graphically in the usual way by drawing the portion of the stress diagram of *Fig.* 89, which is shown to the right of the vector JY, which vector represents the stress developed at joint b in the corresponding bar JY. The diagram of actual stresses at b is shown by the polygon JYZKJ of *Fig.* 89, the side YZ measuring the external or active, and KJ the internal or passive, horizontal stress at b.

The frame, *Fig.* 88, may, in fact, be regarded as being made up of two parts, that above ab, which is loaded with W at c only, and whose stress diagram is shown to right of the vector JY of *Fig.* 89, and the part below ab, which is loaded in addition with the horizontal force H at b, the stress diagram of which is shown to left of vector JY.

The accompanying Table shows the values of the stresses in the several bars of the frame :—

Table of Stresses (*Figs.* 88 *and* 89).*

Outside Flanges,..	AX	CX	EX	GX	IX	LX	NX	OX
Stresses in tons, ..	-18	-52	-76	-90	-78	-45	-13	15·5
Inside Flanges, .	BY	DY	FY	HY	JY	KZ	MZ	OZ
Stresses in tons, ..	+31	+62	+79	+81·5	+91	+94	+65	+38·5
Braces, ..	YA	AB	BC	CD	DE	EF	FG	GH
Stresses in tons, ..	+25	-19	+33	-4	+33	+15	+36	+33 5
Braces, .	HI	IJ	JK	KL	LM	MN	NO	
Stresses in tons, ..	+19	+5·5	+27	-36	+23	-39 5	+7	

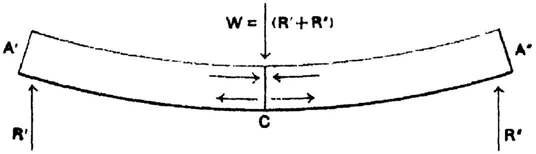
228. *Fig.* 86 shows an arrangement, commonly known as "sheer legs." It is a kind of crane used by builders and erectors, and possesses the advantage of portability. It consists of two main posts, or struts, CF_1 and CF_2, of equal length, the lower ends, F_1 and F_2, of which rest on the ground at a considerable distance apart, and the posts meet together at C, where there is a pulley block, round which a rope passes. The rope is attached to the weight at one end, and the other end is taken to a crab. The sheer legs are prevented from falling forwards by means of a stay or "guy" EC; sometimes by several stays. The stays consist of ropes or chains, which are fixed by means of spikes or pegs into the ground, or are fastened to posts or other convenient objects. They are arranged so as to be easily lengthened or shortened. The posts are thus capable of moving in a vertical plane round the imaginary axis F_1F_2.

If the plane, containing the point of attachment of the load to the rope, viz., C, and the point E, cut the imaginary axis F_1F_2 in the point B, the stresses along the pieces CE, CB can be at once determined, and the stress along CB can then be resolved along CF_1 and CF_2, and thus the resultant stress in each piece of the frame be determined.

* Anglin's "Design of Structures," p 306.

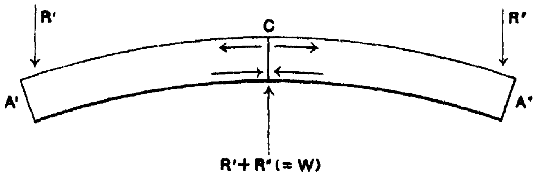
Fig. 81.

Illustrating the analogy between supported Beams and Cantilevers.



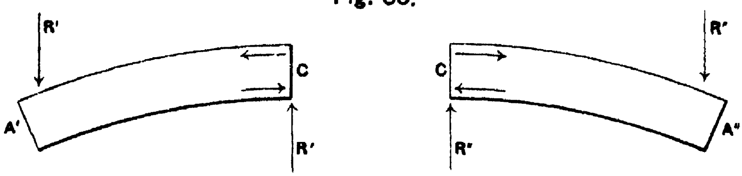
Beam A, *supported at both extremities, and loaded at the middle* C.

Fig. 82.



Beam B, *loaded at both extremities, and supported at the middle* C.

Fig. 83.



The portions A′C, A″C *of Beam* B, *each forming a cantilever, supposed to be fixed at the middle* C, *and loaded with* R′ *at* A,′ *and* R″ *at* A.″

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Fig. 84.

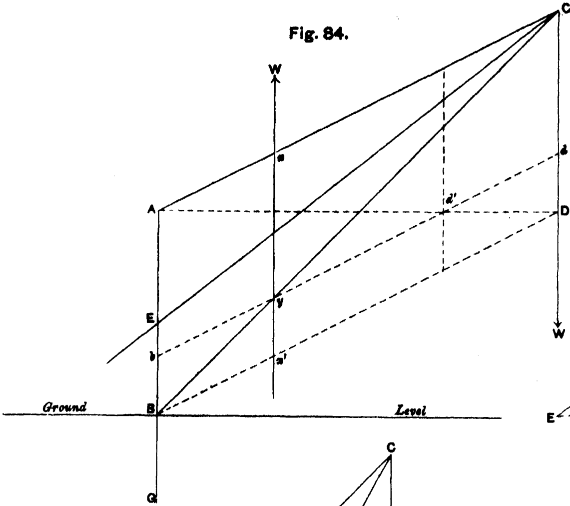

Fig. 86.

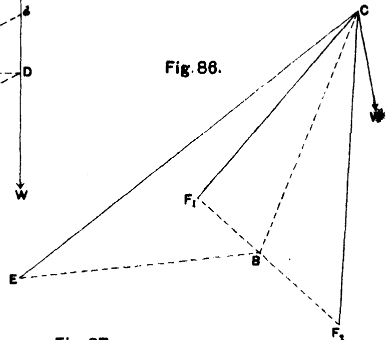

Fig. 85.

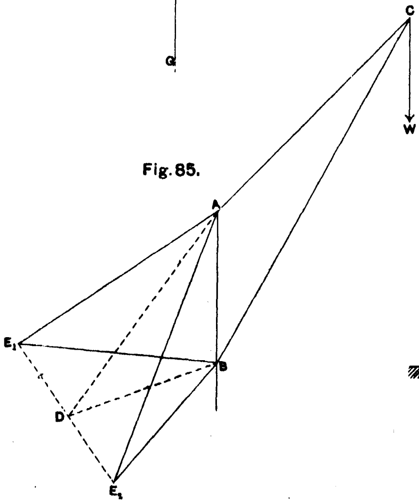

Fig. 87.

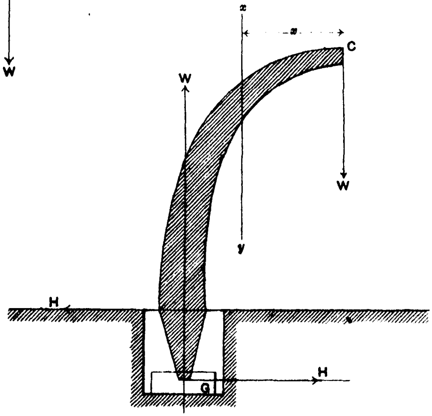

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Fig. 88.

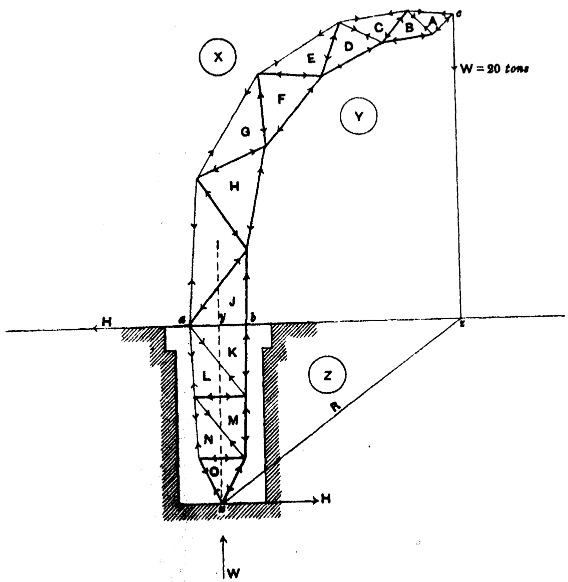
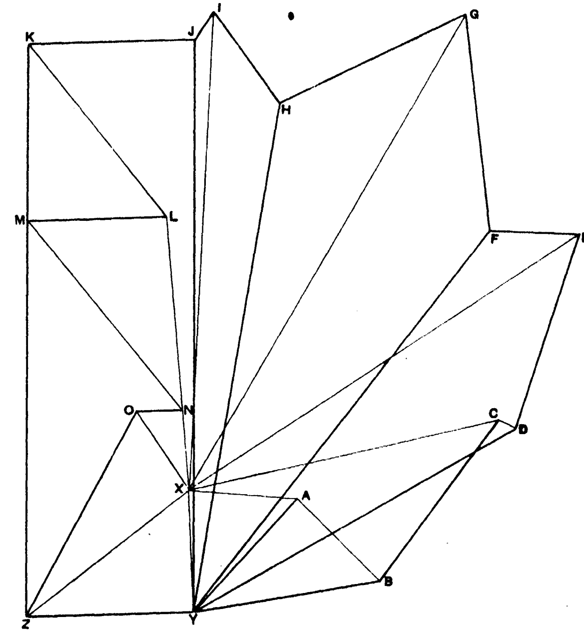


Fig. 89.



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SECTION II.

CHAPTER XV.

RETAINING WALLS—WALLS TO RETAIN WATER.

229.* A Retaining Wall, properly speaking, is a wall built to retain water or an artificial bank of earth, and differs from a Breast Wall, the function of which is often merely to protect a newly exposed surface of earth from the weather, and by so doing to prevent its getting into a condition to require greater support.

230. The dimensions of a breast wall must depend upon a variety of considerations, such as the length of time the newly opened ground is exposed to the action of the weather, the nature of the strata cut through, and their inclination.

Thus in *Fig.* 90, taken from *Dobson's Art of Building*, p. 22, the wall towards which the strata dip, should manifestly be made stronger than the wall on the other side of the cutting, where it need be little more than a thin facing to protect the ground from disintegration.

If the soil cut through consist of clay which has been subjected to great pressure, it swells when exposed to the air, and exercises a force on the back of the wall which is very difficult to estimate. The dimensions of such walls must be determined by experience.

231. In order to calculate the dimensions of retaining walls, the Engineer must know or assume the angle of repose of the earth to be supported, the angle of repose of masonry or brickwork on themselves or each other, and on earth of different kinds, and the weights of earth and masonry.

232. The angle of repose of earth is the angle at which a bank of that particular earth will stand permanently when the cohesion of its particles has been destroyed by the influence of the weather, leaving the

* Paragraphs 229 to 241 are taken, by permission, from Cols. Wray and Seddon's "Instruction in Construction," 3rd Edition, p. 378, *et. seq.*

earth to depend for its stability upon friction alone; and when the state of the earth as regards moisture is the same as that in which the earth behind the wall can, by proper drainage, be ensured to remain.

The angle of repose of masonry on masonry, brickwork on brickwork, &c., is the angle of which the co-efficient of friction of the particular materials is the tangent. This does not take into consideration the resistance of the material to shearing.

233. The average values of the weight of earth and masonry, and the angle of repose and co-efficient of friction of various kinds of earth and masonry, are given below in Tables I., II., and III.; but it is always better to ascertain these values by observation or experiment, where practicable.

TABLE I.

See *Rankine's Useful Rules and Tables* and *Unwin's Lectures on Railway Construction.*

	Co-efficient of friction.	Angle of repose ϕ.	Weight per cubic foot.
Sand, fine and dry, ...	·75 to ·6	37° to 31°	89 lbs. to 118 lbs.
„ wet, ...	·49	26°	
„ very wet, ...	·62	32°	
Vegetable earth, dry, ...	·55	29°	100 lbs. to 120 lbs.
„ „ moist, ..	1 to 1·15	45° to 49°	
„ „ very wet, ..	·3	17°	
„ „ punned, ..	2·25 to 3·49	66° to 74°	
Clay, dry, ...	·55	29°	120 lbs. to 135 lbs.
„ damp, .. / „ well drained, ...	1	45°	
„ wet, ...	·31 to ·25	17° to 1°	
Gravel, clean, ...	1·11	48°	90 lbs. to 110 lbs.
„ with sand, ...	·49	26°	
Loose shingle, ...	·7 to ·8	35° to 39°	
Peat, ...	·25 to 1	14° to 45°	

TABLE II.

See *Rankine's Applied Mechanics*, p. 211.

	f, or tan ϕ.	ϕ.
Masonry and brickwork, dry, ...	·6 to ·7	31° to 35°
" with wet mortar, ...	about ·47	about 25¼°
" with slightly damp mortar, ...	" ·74	" 86½°
" on dry clay, ...	" ·51	" 27°
" on moist clay, ...	" ·33	" 18¼°
Timber on stone, ...	" ·4	" 22°
Iron on stone, ...	·7 to ·3	35° to 16⅔°
Timber on timber, dry, ...	·5 to ·25	26½° to 14°
" " soaked, ...	·2 to ·04	11½° to 2°
" on metal, ...	·6 to ·2	31° to 11⅓°
Metal on metal, ...	·25 to ·15	14° to 8½°

TABLE III.

Brickwork, ordinary stock, in mortar, ..	100 to 110 lbs. per cubic ft.
" " in cement, ..	102 to 112 " "
Masonry, rubble,..	140 " "
" ashlar, limestone, ...	150 " "
" granite, ...	160 to 170 " "
Chalk, ...	117 to 174 " "

234. Retaining walls are always treated as Structures of Uncemented Blocks, and the first determinations necessary are the direction, magnitude, and point of application of the resultant pressure of earth or water which the wall will be called upon to retain.

We shall consider Structures designed to retain pressure from without under two heads—

I. Structures designed to resist Water pressure.

II. Structures designed to resist Earth pressure.

STRUCTURES DESIGNED TO RESIST WATER PRESSURE.

235. The pressure of water, regarded as a perfect fluid and incompressible, which for the purposes of the Engineer it may be taken to be without appreciable error, is (*vide* any Manual on Hydrostatics)—

1st. Directly proportional to the surface on which it presses.

2nd. Directly proportional to the depth of that surface below the surface of the water.

3rd. Normal to the surface against which it presses.

4th. Equal in intensity in all directions.

Thus, let AB (*Fig.* 91) be a plane surface, such as the side of a reservoir, pressed upon by water which it retains; then the pressure on a unit of surface CD is equal to the weight of the column of water CDEF standing on it, the centre of pressure being at the intersection of a vertical through the centre of gravity of the prism CDEF with its base CD; and the direction of the pressure is normal to CD.

For the vertical wall GH (*Fig.* 91), the vertical pressure per unit of surface at the depth GH is equal to the weight of a column of water of the height GH, having a base of one unit of surface; and as water presses equally in all directions, the intensity of the pressure normal to GH, at the depth GH, is given by the weight of the same column of height GH.

Expressing this symbolically, the intensity of the pressure on either the sloping or the vertical wall $= w'h$; where w' is the weight of water per cubic unit, and h the mean depth of the superficial unit.

236. In dealing with water pressure, English Engineers usually take the units of length in feet, and the units of weight in lbs., the weight of water per cubic foot being taken at 62·5 lbs.; its true weight at a temperature of 60° Fahr. being 62·68 lbs. per foot cube.

273. The graphic representation of the pressure in the two cases is, as shown in *Figs.* 92 and 93, the base BC of the right-angled triangle ABC representing $w'h$ in each case, *i.e.*, the depth of water in feet multiplied by the weight of water per cubic foot, on any convenient scale; and as the pressure varies uniformly with the depth, the area of the triangle ABC will represent the whole pressure on the face AB, the resultant of which, passing through the centre of gravity of the figure in a direction normal to AB, fixes the centre of pressure at D, where BD $= \frac{1}{3}$ BA.

238. Since the area of the triangle is measured by $\frac{1}{2}\, w'h^2$, if y be written for h, and x for the corresponding resultant pressure, the locus of the curve, the ordinates of which represent depths and abscissæ corresponding pressures, will be given by the relation $x \propto y^2$. This represents a parabola whose apex is at A and axis coincident with the surface. But it must be remembered that the point of action of the pressure, to which x corresponds, is at depth $= \frac{2}{3}\, y$ below the surface. If the value of y be calculated for one or more depths, the curve can be described by the method explained in para. 181, Vol. I.

239. It is sometimes convenient to treat the vertical and horizontal components of the normal pressure separately, as in *Fig.* 94, the vertical component v being the weight of the prism of water AEB, acting through its centre of gravity at D; and H being the horizontal pressure on the vertical plane EB, acting at D, where BD $= \frac{1}{3}$ BA.

240. When the water presses on a polygonal face, as is sometimes the case in large reservoir dams, the pressure on any side such as CD (*Fig.* 95) may be represented by a trapezium, CEFD, where CE $= w'h$ and FD $= w'h_1$.

The centre of pressure H is found by drawing HG normal to the side CD through the centre of figure G of the trapezium.

In this case the vertical component is the weight of the prism of water CEFD, *Fig.* 96, lying vertically above CD, thus adding to the weight and stability of the wall, the centre of pressure H being at the intersection of the vertical through G, the centre of gravity of the prism, with its base CD. The horizontal component is the difference of the pressures on the vertical planes CE and DF, acting at H, as represented by the figure lkDm, and tending to overturn the wall.

241. When the surface on which the water presses is curved, the pressure may be found by replacing the curve by a polygon having a large number of sides, and dealing with each side of the polygon as in the last case.

The Conditions of Stability of a Reservoir Wall or Dam.

242. In the following pages the design of a Wall or Masonry Dam, required to resist the pressure of water or other fluid *on one side only*—such, for instance, as might be built across a valley to dam the course of a river, and maintain the surface of the water above its natural level—will be considered. Such a wall, being regarded as a Structure of Uncemented Blockwork, can afford the necessary resistance in virtue of its weight alone, and its profile, therefore, must be designed so as to fulfil the conditions of stability referred to in Chapter X.; its horizontal trace depending, as a rule, on considerations other than those of stability, such, for instance, as the form of the ground surface or valley, &c.

In the *Engineer* of January 5th, 1872, the copy of a Report by Professor Rankine on the Design and Construction of Masonry Dams was

published; a portion of it is reproduced in Appendix B. From para. 11 of this Report it will be seen that Professor Rankine considers "that there ought to be no practically appreciable tension at any point of the masonry, whether at the outer face when the reservoir is empty, or at the inner face when the reservoir is full." This condition is evidently of primary importance on the inner face, where any penetration of water into an open joint would decrease, by its upward pressure, the stability of the dam. It has been already shown (para. 193) that this condition is secured by so designing the profile that the lines of resistance shall in all cases—that is, when the reservoir is full as well as when it is empty—lie within the middle third of the wall's thickness.

As the depth of the wall, then, measured downwards from its top as origin increases, so also must its horizontal thickness, in order to fulfil the necessary conditions of stability and strength at each horizontal joint, and it will be seen from para. 6 of Professor Rankine's Report, that he considers that "the limit of the safe intensity of vertical pressure should decrease in some proportion as the inclination of the wall's surface to the vertical, or its *batter*, increases. For, the direction in which the pressure is exerted amongst the particles close to either face of the masonry is necessarily that of a tangent to that face; and, unless the face is vertical the vertical pressure found by means of the ordinary formula is not the whole pressure, but only its vertical component; and the whole pressure exceeds the vertical in a ratio which becomes the greater, the greater the batter, or deviation of the face from the vertical. The outer face of the wall has a much greater batter than the inner face; therefore, in order that the masonry of the outer face may not be more severely strained when the reservoir is full than that of the inner face when the reservoir is empty, a lower limit must be taken for the intensity of the vertical pressure at the outer than at the inner face." In paras. 9 and 13 of the Report the proposed limits are stated; they should be such that the vertical pressure per square foot at the inner face shall never exceed that of 160 cubic feet weight of masonry, which, for ordinary brickwork, amounts to about 20,000 lbs; and that at the outer face the weight of 125 cubic feet of masonry, or about 15,625 lbs. per square foot for ordinary brickwork, "at the point where it is most intense, and to diminish in going down from that point."

It will be sufficient for all practical purposes to regard an expression

of the following form as a symbolical statement of the law of strength of the wall, *viz.*, limit of vertical pressure not to exceed $S\left(\frac{1+\cos\chi}{2}\right)$ per square foot, where χ represents the angle the tangent to the wall's face, or the *batter*, makes with the vertical, and S stands for the limiting intensity of pressure in pounds per square foot, when the face is vertical. This subject will be treated more fully later on (para. 252).

243. *The Conditions of Stability of a Reservoir Wall or Dam* may be briefly stated as follows:—

(1). The resultant of all the forces acting at any horizontal joint must fall within the middle third of the thickness of the dam measured at that joint.

(2). The stresses in the faces of the dam must not exceed the safe limit.

(3). These two conditions must be satisfied for the two cases of both reservoir full and reservoir empty.

A fourth condition might be added, *viz.*, that at every bed joint the dam must be safe against sliding; but this condition is practically always fulfilled when conditions (1) and (2) are satisfied.

244. Before the design of the profile can be commenced, it is necessary to determine the width of the wall's top, the case when it is possible to have a feather edge seldom, if ever, occurring, it being generally necessary for some reason—such, for instance, as to resist the shocks of waves in a large reservoir, or to afford a passage either for carts or foot passengers, or for the transport of materials for repairs—to give some considerable thickness to the wall at top; which top surface is, moreover, for purposes of safety usually made flush with that of the highest water level. Failing any determining considerations, however, such as those alluded to, Mr. Courtney in Paper No. 2110, Vol. LXXXV. of the "Proceedings of the Institute of Civil Engineers" gives the following empirical formulæ:—

If b_1 represent the width of the wall's top,

d_0 its height above highest water level,

H, the total height of the wall,

Then

$$b_1 = 4\cdot0 + 0\cdot07\ \text{H}, \qquad (1).$$

$$d_0 = 1\cdot8 + 0\cdot05\ \text{H}, \qquad (2).$$

Design of the Wall's Profile.

245. The width of wall at top having been thus decided on, its profile may be designed in four portions, as follows :—

Portion I.—It is evident that the upper portion of the wall's profile may be rectangular, and that there is a certain depth d_1, *Fig.* 96*a*, at which condition (1), which limits the centre of pressure to the middle third of the wall's thickness, is just fulfilled for the full reservoir, and more than fulfilled for the empty reservoir; the same condition being more than fulfilled at all depths less than d_1.

Portion II.—Below depth d_1 it is usual to batter the outer face while retaining the inner face vertical, and it will be found that at a certain horizontal joint of this portion of the wall, condition (1) is *just* fulfilled for the full, as well as for the empty, reservoir wall.

Portion III.—Below this horizontal joint, therefore, it becomes necessary to batter the inner, or wetted, face as well as the outer, or dry, face, in order to retain the line of pressure within the middle third of the wall's thickness when the reservoir is empty.

Portion IV.—This due fulfilment of the first condition of stability for the dam's wall when the reservoir is full as well as when it is empty, will be found to include also the fulfilment of the second condition down to a certain horizontal joint of Portion III., below the depth of which it will be found necessary to increase the batter of *both* faces in order to retain the maximum intensity of stress within the prescribed safe limits.

Each of the four portions will now be dealt with separately.

Design of Portion I.

246. Supposing b_1, the breadth of the wall's top, and d_0, its height above highest water level, to be known, let ABED, *Fig.* 96*a*, represent a cross section of the rectangular portion. Then, if W represent the weight of ABED, acting vertically downwards through its centre of gravity and bisecting DE in q, the thickness of the wall (measured at right angles to the plane of the cross section) being supposed equal to unity; and if

P represent the resultant horizontal fluid pressure, acting over the vertical area of wall of height d_1 and width unity, at a depth $= \frac{2}{3}d_1$ below the fluid surface;

w, the weight of a cubic foot of masonry;
w', that of a cubic foot of the fluid contained in the reservoir.

Then, $W = wb_1 (d_0 + d_1)$, and $P = \frac{w'}{2} \times d_1^2$.

If the line of action of W meet that of P in p, and the horizontal surface of the joint DE in q; and if r be the point in which the resultant of P and W combined meet the surface DE, we have

$$\frac{qr}{pq} = \frac{P}{W} = \frac{w'd_1^2}{2wb_1(d_0 + d_1)}$$

$$\text{whence } qr = pq \frac{w'd_1^2}{2wb_1(d_0 + d_1)}$$

But $pq = \frac{1}{3}d_1$, and qr in the limiting position imposed by condition (1) must not exceed $\frac{1}{6}b_1$. Putting $\frac{w}{w'} = \theta$, we have for the limiting position the relation

$$d_1^3 - \theta b_1^2 (d_1 + d_0) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

If, however, as is usually the case, *the surface of the fluid in the reservoir be supposed to be flush with the top of the dam, so as to provide against a possible rise due to a flood, or an increase of pressure due to waves*, then d_0 will vanish from equation (3) and the relation becomes

$$d_1 = b_1 \sqrt{\theta}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

This value of d_1 may be either calculated numerically, or determined geometrically by the following simple construction:—

Draw CB and BD, *Fig.* 96*c*, at right angles to one another, taking $CB = w'$ on some convenient scale of loads, and $BD = b_1$ on the lineal scale; join CD, and draw DE at right angles to it to meet CB produced in E; then evidently $BD = \sqrt{CB \times BE}$. Now take BA $= w$, on the same scale of loads as $BC = w'$, and on AE describe the semicircle AGE, producing BD to meet it in G; then, evidently

$$BG = \sqrt{AB \times BE}, \text{ but } BD = \sqrt{CB \times BE}$$

$$\therefore BG = BD\sqrt{\frac{AB}{CB}} = b_1 \sqrt{\theta} = d_1$$

Below depth d_1, the horizontal width of the cross section must be continually increased as the depth of the wall, measured from its top as origin, increases, in order to satisfy condition (1), and for this portion it will be found best to calculate an approximate profile by the simple formulæ to be given presently, and test its stability by graphic methods already described, altering the calculated profile, if necessary, accordingly.

Design of Portion II.

247. The increase in horizontal width for increased depth of wall below the fluid surface, required to fulfil condition (1) for both reservoir full and reservoir empty, may be calculated for a series of zones, whose bases are horizontal, on the supposition that the outer or dry face of the wall is polygonal, and the inner, or wetted, face vertical; moreover, by reference to the profiles given in *Plates* XXII. and XXIII., it will be observed that the outline of the dry face is such that but little difference in the section results from supposing the top joint DE to be common to all the zones, their heights and width of base alone varying; that is, from supposing the extremities D and E of the base of Portion I. to be joined by straight lines to the extremities M and Q respectively of any zone base QM, as shown in *Fig.* 96*b*; any difference being on the side of safety, as will be proved numerically. The more exact expression will, however, first be formed, in order that the numerical comparison may be made, and the simpler form will then be deduced from it. For reasons of simplicity and safety, moreover, *the surface of the fluid will in each case be supposed flush with the top of the dam.*

The following method yields a result which may be made to differ as little as is desired from the true one by sufficiently diminishing the zone height.

Let GF, *Fig.* 96*b*, be the lowest joint whose proper width b is known; W' the weight of the structure lying above GF; the distance c of its line of action from the vertical through B, that is, from the wetted face of the wall, being supposed known; let a be the vertical height of the several zones, that is, the vertical distance apart of the extreme zone joints, which may be made as small as we please; and x the required width of the zone base LK next below GF. Then we have for the total weight T of the structure lying above LK

$$T = W' + w\left\{ab + \frac{a(x-b)}{2}\right\} = \tfrac{1}{2}\left\{2\,W' + wa\,(x+b)\right\}$$

and for the distance $\bar{c}$ of its line of action from the vertical through B,

$$\bar{c} = \frac{Wc' + w\left\{\dfrac{ab^2}{2} + \dfrac{a}{6}(x-b)(x+2b)\right\}}{\tfrac{1}{2}\left\{2\,W' + wa\,(x+b)\right\}}$$

$$= \frac{1}{3}\left\{\frac{6\,W'c + wa\,(x^2 + bx + b^2)}{2\,W' + wa\,(x+b)}\right\}$$

It will simplify the calculations if we write A for the area of the whole

section of the wall lying above joint GF, so that $W' = Aw$. The above equations then become

$$T = \frac{w}{2}\left\{2A + a\,(x + b)\right\}, \dots\dots\dots\dots (5).$$

$$\bar{c} = \frac{1}{3}\left\{\frac{6Ac + a\,(x^2 + bx + b^2)}{2A + a\,(x + b)}\right\}, \dots\dots\dots\dots (6).$$

If d be the depth of LK below the highest assumed water surface, we have for the resultant fluid pressure P', acting over the vertical area of the wall, whose depth is d and width unity, the expression $P' = \frac{w'}{2}\,d^2$ the centre of which pressure is at a depth $= \frac{2}{3}\,d$ below the water surface, that is, at a point situated $\frac{1}{3}d$ vertically above LK.

If, as before, we suppose the line of action of P', (which, under the assumption that the wetted face of the wall is vertical, acts horizontally) to meet that of T in the point p; the line of action of T to meet the joint's surface in q; and that of P and T combined to meet it in r, *Fig.* 96*b*, then will

$$\frac{rq}{pq} = \frac{P'}{T} = \frac{\frac{w'}{2}\,d^2}{\frac{w}{2}\left\{2\,A + a\,(x + b)\right\}} = \frac{1}{\theta}\left\{\frac{d^2}{2\,A + a\,(x + b)}\right\}$$

$$\therefore\; rq = \frac{d}{3} \times \frac{1}{\theta}\left\{\frac{d^2}{2\,A + a\,(x + b)}\right\} = \frac{d^3}{3\theta}\left\{\frac{1}{2\,A + a\,(x + b)}\right\}$$

In order to fulfil condition (1), $\bar{c} + rq$ must not exceed $\frac{2}{3}\,x$, so that in the limiting position we have

$$6\,Ac + a\,(x^2 + bx + b^2) + \frac{d^3}{\theta} = 2x\left\{2A + a\,(b + x)\right\}$$

$$\therefore\; x^2 + x\left(4\,\frac{A}{a} + b\right) = 6\,\frac{A}{a}\,c + b^2 + \frac{d^3}{a\theta}$$

and

$$x = \left\{4\left(\frac{A}{a}\right)^2 + 2\left(\frac{A}{a}\right)(b + 3c) + \frac{5}{4}\,b^2 + \frac{d^3}{a\theta}\right\}^{\frac{1}{2}} - \left(2\,\frac{A}{a} + \frac{b}{2}\right), (7),$$

in which expression a may be made as small as we please; the smaller a is taken, the more nearly will the polygonal form of the dry face approach a curved one, and the more accurate, therefore, be the result.

247a. In order to arrive at a numerical result for purposes of comparison, let a be reckoned in terms of d_1. Thus, if $a = \nu d_1$, where ν is any positive number, integral or fractional, then $d = (1 + \Sigma\nu)\,d_1$, where $\Sigma\nu d_1$ stands for the sum of terms similar to νd_1; the last term under the radical sign of equation (7), remembering that $d_1 = \sqrt{\theta}\,b_1$ by equation 4), then becomes

$$\frac{d^3}{a\theta} = \frac{(1 + \Sigma\nu)^3}{\nu d_1 \times \theta} d_1^3 = \frac{(1 + \Sigma\nu)^3}{\nu} b_1^3$$

Equations (6) and (7) may then be written

$$\bar{c} = \frac{6\ \mathrm{A}c + \frac{\nu}{2}(x^2 - bx + b^2)\, d_1}{3\left\{2 \times \mathrm{A} + \frac{\nu}{2}(x + b)\, d_1\right\}} \quad \text{.........................(8).}$$

$$x = \left[\frac{\mathrm{A}}{\nu d_1}\left\{4\left(\frac{\mathrm{A}}{\nu d_1}\right) + 2\,(b + 3c)\right\} + 1{\cdot}25b^2 + \frac{(1 + \Sigma\nu)^3}{\nu} b_1^2\right]^{\frac{1}{2}}$$
$$- \left\{2\left(\frac{\mathrm{A}}{\nu d_1}\right) + \frac{b}{2}\right\} \quad \text{......................... (9).}$$

For purposes of comparison it will be sufficient to deal with the zones shown in *Figs.* 97*a* and 97*d*, *Plate* XXIII., whose heights are equal to d_1; the values of x and $\bar{c}$ for the first four *half* zones will, however, be also calculated.

If, then, ν be put $= \frac{1}{2}$, we have, in equations (8) and (9), for the first half zone, lying immediately below Portion I.,

$$\left.\begin{aligned} \Sigma\nu &= \tfrac{1}{2} \\ \mathrm{A} &= \mathrm{A}_1 = b_1 d_1 \\ b &= b_1 \\ c &= \tfrac{1}{2} b_1 \\ d &= \tfrac{3}{2} d_1 \end{aligned}\right\} \quad \begin{aligned} &\text{whence } \bar{c} = 0{\cdot}520\ b_1 \\ &\text{and } \quad x = 1{\cdot}330\ b_1 \end{aligned}$$

If, again, $\nu = \frac{1}{2}$, we have for the second half-zone

$$\left.\begin{aligned} \Sigma\nu &= 1 \\ \mathrm{A} &= \mathrm{A}_1 + \tfrac{1}{2}(1 + 1{\cdot}33)\, b_1 \times \tfrac{1}{2} d_1 \\ &= 1{\cdot}5825\ b_1 d_1 \\ b &= 1{\cdot}33\ b_1 \\ c &= 0{\cdot}52\ b_1 \\ d &= 2\ d_1 \end{aligned}\right\} \quad \begin{aligned} &\text{whence } \bar{c} = 0{\cdot}604\ b_1 \\ &\text{and } \quad x = 1{\cdot}755\ b_1 \end{aligned}$$

In this way the values of $\bar{c}$ and x corresponding to increasing values of d may be determined in terms of b_1, as shown in the following Table :—

1	2	3	4
$d \div d_1$.	$c \div b_1$.	$x \div b_1$.	$x \div 3b_1$.
$1\frac{1}{2}$	0·520	1·330	0·443
2	0·604	1·755	0·585
$2\frac{1}{2}$	0·723	2·248	0·747
3	0·869	2·766	0·922
4	1·077	3·821	...
5	...	4·749	...

From columns 2 and 4 of the above Table it is evident that the batter of the inner face commences at some depth greater than $2d_1$ and less than $2\frac{1}{2}d_1$, *vide* para. 250.

248. Now, equations (8) and (9) are troublesome to apply, and, moreover, any error made is cumulative, so that since, as will be seen, a sufficiently accurate result may be obtained under the hypotheses mentioned at the commencement of para. 247, we shall, as already explained, suppose the top joint DE, *Plates* XXII. and XXIII., to be common to all the zones, their heights and bases only varying, moreover, the value of d will be reckoned in terms of d_1; thus, if $d = nd_1$, where n is either a positive integer or fraction and reckoned from the base of Portion I., then, in equations (6) and (7),

$$a = nd_1,\ b = b_1,\ \mathrm{A} = b_1 d_1, \text{ and } c = \tfrac{1}{2}b_1$$

d_1 disappears entirely from these equations, and they become

$$\bar{c} = \tfrac{1}{3}\left\{\frac{3b_1^2 + n(b_1^2 + b_1 x + x^2)}{2b_1 + n(b + x)}\right\} = \tfrac{1}{3}\left\{\frac{(x + b_1)nx + (n+3)b_1^2}{nx + (n+2)b_1}\right\}, \quad (10),$$

$$x = b_1\left[4\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left\{5 + (n+1)^3\right\} + \tfrac{5}{4}\right]^{\frac{1}{2}} - \tfrac{1}{2}b_1\left(1 + \frac{4}{n}\right), \quad (11),$$

$$\text{or } x = \frac{1}{2n} \times b_1\left\{(4n^4 + 12n^3 + 17n^2 + 24n + 16)^{\frac{1}{2}} - (n+4)\right\}, \quad (12),$$

any one of which expressions is easy of application.

The values of x corresponding to successive values of n may now be expressed directly in terms of b_1, thus—

1	2	3	4
$d \div d_1$.	n.	$x_n \div b_1 = k$.	$(x_n - x_{n-1}) \div b_2$.
$1\frac{1}{2}$	$\frac{1}{2}$	1·330	...
2	1	1·772	0·442
$2\frac{1}{2}$	$1\frac{1}{2}$	2·262	0·491
3	2	2·772	1·000
4	3	3·803	1·031
5	4	4·831	1·028
6	5	5·854	1·023
7	6	6·871	1·017

so that the relation exhibited in column 3 may be written $x = kb_1$, where k has the above numerical values for the particular value of n chosen.

From column 4 it is evident that between the values $n = 2\frac{1}{2}$ and $n = 4\frac{1}{2}$ about, that is, from depth $d = 3{\cdot}5d_1$ to depth $5{\cdot}5d_1$, about, the surface of the outer face of the wall is nearly a plane, whose inclination to the vertical, or batter, is approximately equal to $\tan^{-1}\left(1{\cdot}03\,\frac{b_1}{d_1}\right)$, that is, to $\tan^{-1}\left(\frac{1{\cdot}03}{\sqrt{\theta}}\right)$, since $d_1 = \sqrt{\theta} \times b_1$ by equation (4).

248a. For purposes of comparison the values of $x \div b_1$ as given by equations (9) and (12), are shown in the accompanying Table, and it will be observed, since the several values of x vary only with that of b_1, that they are the same for *all* profiles, of whatever material the wall be built and whatever fluid the reservoir contains, *provided only that the width of the walls be the same at top*; and that such profiles differ only in regard to d_1, which is a function of θ, being more elongated the greater θ is. Profiles of walls, therefore, of the same top width, but built of different material or to retain different fluids, may be regarded as projections by parallel rays of some ideal profile, the inclination of the projecting rays differing according to the value of θ, which, were the fluid distilled water, would represent the specific gravity of the material of which the dam is constructed.

The values of x given in the accompanying Table are those usually comprised in Portions II. and III.

$d \div d_1$.	Values of $x \div b_1$.		Difference.
	By Equation (9).	By Equation (12)	
$1\frac{1}{2}$	1·330	1·330	...
2	1·755	1·772	+0·017
$2\frac{1}{2}$	2·248	2·262	+0·014
3	2·766	2·772	+0·006
4	3·821	3 803	−0·018
5	4·749	4·831	+0 082

It will be observed from the above Table that the profile, as designed

by equation (12) is, with exception of a short portion lying at a depth of about $4d_1$ below the wall's top, slightly broader than that designed by equation (9); and with regard to this portion it must be borne in mind that no account has been taken of the weight of masonry making up the batter of the inner face of the wall, nor of the obliquity with regard to the horizon of the fluid pressure on that face, both of which causes tend to throw the centre of pressure of the full reservoir dam further towards the inner face of the wall. It may, therefore, be concluded that equations (10) to (12), while yielding a profile sufficiently accurate in form, possess the advantage of great simplicity as compared with equations (6) and (7). In para. 250*a* the stability of the structure at depth $d = 4d_1$ is examined.

249. Equation (11) or (12) determines the profile of a dam-wall, whose line of resistance cuts every horizontal joint at a distance of two-thirds its thickness, measured from the wall's inner or wetted face, the reservoir being supposed full; and equation (10) determines the locus of the centres of pressure of successive horizontal joints, *i.e.*, the line of resistance, of the same structure, when the reservoir is empty. If, then, c be put equal to $\frac{x}{3}$, and the two equations solved for simultaneous values of x and n, the horizontal joint will be known at which the first condition of stability is simultaneously satisfied for the case of reservoir full as well as that of reservoir empty. Equation (10) under these conditions reduces to the form

$$x = \tfrac{1}{2}b_1 (n + 3), \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

the locus of which, since n is, by hypothesis, a continuously varying quantity, is a straight line, which may be described by producing the vertical drawn through B upwards to C, *Figs.* 96*b*, 97*a* and 97*d*, making $BC = 2BE = 2d_1$, joining CA and producing it.

Thus, values of x corresponding to given values of n, the increments of which may be made as large or as small as we please, being calculated and plotted, the point K, *Fig.* 96*b*, in which the straight line represented by equation (13) meets the polygon thus represented by equation (11) or (12), determines the horizontal joint LK at which condition (1) of para. 243 is just satisfied when the reservoir is full as well as when it is empty. The batter might, indeed, be commenced rather above the joint given by this geometrical construction for reasons of safety, *vide* para. 250.

It will be observed that the smaller the increments of n be taken the

more nearly will the form of the outer face approach a curved rather than a polygonal one; also that the distance nd_1 is measured downwards from the point E and corresponds to depth $d = (n + 1) d_1$ measured from B.

From the wall's top down to this joint the inner, or wetted, face is built vertical; below this joint, a batter must be given to it in order to retain the line of resistance of the empty reservoir wall within the middle third of its thickness.

N.B.—Equations (12) *and* (13) *might, of course, be at once written down without previous formation of the more exact forms expressed by equations* (8) *and* (9).

Design of Portion III.

250. By equating equations (12) and (13) we have

$$4n^4 + 12n^3 + 17n^2 + 24n + 16 = \{n (n + 3) + (n + 4)\}^2,$$

and, by multiplying out in the usual way we obtain the relation

$$(n^2 - 2) (3n + 4) - n = 0$$

one positive root of which is evidently greater than 1·470 and less than 1·475. At the horizontal joint, therefore, at which the inner face of the dam should *just* cease to be vertical n is rather less than 1·475 d_1; it would be *safe* to make it $1{\cdot}4d_1$ or $1{\cdot}3d_1$; the former value will be adopted. At a depth, therefore, of $2{\cdot}4d_1$ (or $2{\cdot}4 \times \sqrt{\theta} \times b_1$) below the wall's top, the batter of the inner face should commence. Below this depth the profile of the outer face may still be designed by means of equation (11) or (12), values of x being measured, as before, from the vertical passing through B, the batter of the inner face being so slight as to add but little to the weight of the wall. For the same reason the hypotheses mentioned at the commencement of para. 247 may still be observed; so that if $(n - 1{\cdot}4) d_1$ be the height of zone KLNM, lying immediately below joint LK, *Fig.* 96*b*, whose base NM is made up of QM $= x$ and QN $= x'$, of which x is given by equation (11) or (12) and corresponds to depth $y = (n + 1) d_1$, and x' is the required batter base of the inner face, then may the magnitude of x' be determined by taking moments about Q in the usual way, thus—

If W′ be the weight of the structure lying above joint NM, and $\bar{c}'$ the distance of its line of action from Q, then

$$W' = w \{\tfrac{1}{2} (n - 1{\cdot}4) d_1 x' + (n - 1) d_1 b_1 + \tfrac{1}{2} n d_1 (x - b_1)\}$$
$$= \tfrac{1}{2} w d_1 \{(n - 1{\cdot}4) x' + nx + b_1 (n + 2)\}.$$

In taking moments about Q, x' must be regarded as opposite in sign to x, so that

$$W'\bar{c'} = \tfrac{1}{2}\, wd_1 \left[-(n-1\cdot4)\,\tfrac{1}{3}\,(x')^2 + (n+1)\,b_1^2 + n\,(x-b_1)n \left\{ b_1 + \tfrac{1}{3}\,(x-b_1) \right\} \right]$$

$$= \tfrac{1}{6}\, wd_1 \left\{ -(n-1\cdot4)\,(x')^2 + n\,(x^2 + b_1x + b_1)^2 + 3b_1^2 \right\}$$

substituting for W′ and putting $\bar{c'} = \frac{1}{3}\,(-x' + x)$ we have

$$x' = \frac{b_1\,\{2x - (n+3)\,b_1\}}{1\cdot4x + (n+2)\,b_1}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

or, substituting for x the value kb_1, in which k has the proper numerical value corresponding to that of n, as given in the Table of para. 248, we have

$$x' = \left\{ \frac{2k - (n+3)}{1\cdot4\,k + (n+2)} \right\} b_1, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

The values of x' corresponding to those of x, or kb_1, and n can thus be tabulated—

$d \div d_1$.	n.	$k = x \div b_1$.	$k' = x' \div b_1$.
3	2	2·772	0·069
4	3	3·803	0·156
5	4	4·831	0·208
6	5	5·854	0·244

250a. It can now be shown at once that, although the direction of the fluid pressure on the inner battered face of the dam be still supposed horizontal, the first condition of stability is amply satisfied for the full reservoir wall, designed by equation (12) at depth $d = 4d_1$.

For, referring to *Fig.* 96*b*, and to the Table given in para. 248, it is easy to show that

$$\left.\begin{aligned} &\text{If } n = 3 \\ &x = 3\cdot803b_1 \\ &x' = 0\cdot156b_1 \\ &d = 4d_1 \end{aligned}\right\} \begin{aligned} &\text{The area of QBADM} = 7\cdot9125b_1d_1 \\ &\quad\text{and that of LNQ} = 1\cdot248b_1d_1 \end{aligned}$$

whence, in *Fig.* 96*b*, the total weight borne on the joint NM

$$= T' = 9\cdot1605b_1d_1 \times w,$$

the resultant horizontal fluid pressure $= P'' = \frac{1}{2}\,(4d_1)^2 \times w'$, whence, if δ be the distance of the centre of pressure of the full reservoir wall

from N, we have, as in para. 247 (remembering that T′ acts at $\frac{1}{3}$NM from N),

$$\frac{\delta - \frac{1}{3}(x' + x)}{\frac{1}{3}d_1} = \frac{P''}{T'}$$

whence, by substituting the numerical values, we have

$$\delta = 2\cdot4841b_1.$$

Now the total width of the joint NM $= x' + x = 3\ 959b_1$; hence the limiting value of $\delta = \frac{2}{3} \times 3\cdot959b_1 = 2\cdot6393b_1$, so that there is a difference of $0\cdot155b_1$ on the side of safety.

Graphical Constructions for drawing in the Lines of Resistance.

251. The dimensions of the profile up to this point are, as a rule, best determined by calculation; by the following geometrical methods the lines of resistance may be drawn in graphically; they are usefully applied in Part IV.

(*a*). *To draw the Line of Resistance of the Empty Reservoir.*—As fully explained in paras. 87 and 88 of Chapter VII., let the positions of the centres of gravity of the zones I., II., III.,............, *Fig.* 97, into which the profile has been divided, be denoted by the letters g_1, g_2, g_3............, and through these points draw the vertical load-lines g_1w_1, g_2w_2, g_3w_3 ....; set off the weights Ow_1, w_2, w_3..............w_7w, of the successive zones, commencing from the top, to any convenient scale, and on any convenient load-line Ow, *Fig.* 97*b*; and with any convenient pole P describe the stress diagram POw, and at a convenient distance above or below the profile describe the equilibrium polygon 1 2 3 4 5 6........, *Fig.* 97*c*, with its angles lying in the lines of action of the corresponding loads, so that sides 1 and 2 balance the load line through g_1, that is, balance load w_1, sides 2 and 3 that through g_2, and so on; and by the method fully described in para. 89, determine the lines of action, shown in dotted lines, of the combined loads $w_1 + w_2$, $w_1 + w_2 + w_3$, $w_1 + w_2 + w_3 + w_4$.........., (that is, by producing the sides of the polygon 1 2 3 4.............. successively to intersect side 1); then will the points c_1', c_2', c_3'............ in which the verticals through these points of intersection, being the lines of action of the several combined loads, successively meet the corresponding horizontal base-joints on which they rest, be the centres of resistance of those joints, and the line joining them be, therefore, the

locus of those centres, or the Line of Resistance of the Empty Reservoir. It will be observed that, whereas this line of resistance is inclined to the vertical, the direction of the resultant loads on the several joints is vertical. This construction is applied to the bottom three zones only of *Fig.* 97*a*.

(*b*). *The resultant horizontal fluid pressure acting on each zone or system of combined zones* may be represented graphically in the following manner; this pressure, if the surface of the zone exposed to the action of the fluid be vertical, is the whole resultant fluid pressure, and if inclined to the vertical, is the horizontal component of the whole resultant fluid pressure, enabling the latter to be determined by combining this horizontal component with the weight of the fluid superincumbent on the exposed surface, *vide* para. 257(*c*).

Calculate the resultant horizontal fluid pressure acting over a vertical surface of the wall of width unity and depth equal to that of the lowest horizontal joint of the profile, and draw in the curve of horizontal pressures, which will be a parabola, para. 238; the magnitude of the resultant horizontal fluid pressure acting over the exposed surface of any block or combination of blocks, or at any depth, is then at once known by measurement from the curve. If, for instance, the resultant horizontal pressure P_7', acting over the depth BS, *Fig.* 97*a*, were represented by the straight line Sπ, and the parabola Bπ drawn in, then would the resultant horizontal pressure acting over blocks I. to V. be represented by P_5', and that over block VI. by the difference $P_6' - P_5'$. It will be remembered that each of the resultant pressures P'_1, P_2', P_3'......... acts at $\frac{2}{3}$ the depth of the lowest horizontal joint to which it corresponds.

This construction is specially applicable to Method I., following (*Fig.* 97*a*).

(*c*). *The total resultant fluid pressure acting over the exposed surface of any zone, and its line of action* may be graphically determined as follows:—

At the foot S of the vertical through B, *Fig.* 97*d*, set off $Sl = m \times w'h$ = the intensity of fluid pressure at depth BS = h, multiplied by some convenient multiple m; and join lB. It is necessary to employ the multiple m = 10 or 20 as the case may be, as otherwise the quantity $w'h$ would be too small to plot. In the example given, *Fig.* 97*d*, m is taken equal to 20. Since the intensity of pressure varies directly as the

depth, the abscissæ of the figure BSl, drawn parallel to Sl, measure the intensities of pressure at the corresponding depths. For instance, tv measures the intensity of pressure at the depth B$t = h'$, on the same scale as Sl measures that at depth BS; moreover, the area of the figure lying above any abscissa, as Btv, measures the resultant horizontal fluid pressure acting over a surface of width unity and extending to the depth Bt, the line of action of such resultant fluid pressure passing through the centre of gravity of the representative triangle Btv. Similarly the area of the trapezoid tSlv, lying between the two abscissæ tv and Sl, measures the resultant horizontal fluid pressure acting over the corresponding exposed surface t'S$'$, its line of action passing through the centre of gravity of the trapezoid. And since the intensity of fluid pressure is uniform at the same depth and acts always at right angles to the surface exposed to it, if these abscissæ be set off at their proper depths, as $t'v'$ and S$'l'$, at right angles to an inclined surface, as t'S$'$, immersed in the fluid, then will the Theorem be equally true of the transformed trapezoid t'S$'l'v'$. The Theorem is illustrated in *Fig.* 97d, and requires no further comment. It is specially applicable to Method II. following.

The area of the representative trapezoid may be either calculated, or determined graphically as follows:—

If in *Fig.* 97d, S$'l'$ measure $m \times w'h$ weight units, and $t'v'$ measure $m \times w'h'$ weight units, and z' length units be the distance between S$'l'$ and $t'v'$, that is, the length of S$'t'$, then will the area of the trapezoid $v'l'$S$'t'$ be measured by $\frac{z'}{2} \times w'\ (h + h')$ weight units $=$ $\frac{z'}{2m} \times mw'\ (h + h') = \frac{z'}{2m}\ (\text{S}'l' + t'v')$. As in *Fig.* 98, take $ca =$ $\frac{z'}{2}$ and $cb = m$, each in units of *length*, and inclined to one another at any convenient angle acb; take $cd = v't' + \text{S}'l'$, and draw de parallel to ba; then $ce = cd \times \frac{ca}{cb} = (v't' + \text{S}'l') \times \frac{z'}{2m} = \frac{z'}{2} w'(h + h')$.

(d). *To draw the Line of Resistance of the Full Reservoir.*—The line of resistance of the full reservoir may be drawn in by either of the following methods:—

Method I.—By combining the resultant fluid pressures acting on systems of successively combined zones with the already determined

weights of the latter, as in *Fig.* 97*a*. The line of action of the resultant weight of any system of combined zones being known from section (*a*), page 130, it is only necessary to draw through the point, in which it is cut by the line of action of the corresponding resultant fluid pressure, a straight line parallel to the resultant of these two forces, as given by the stress-diagram. The point in which this latter resultant line cuts the horizontal joint on which the combined zones stand is the centre of pressure of that joint. In this way the line of resistance of the wall, when the reservoir is full, may be drawn in. *This method is applied to the bottom three zones only of the profile shown in Fig.* 97*a.*

Method II.—By the general method described in para. 173, that is, by determining the point in which the resultant line of action of the combined loads acting on each zone separately meets the corresponding zone base, or horizontal joint on which the zone stands, and joining the points so found, as in *Fig.* 97*d*. In applying this method it is only necessary to be careful to see that that resultant line is selected on the stress-diagram which belongs to the forces under consideration, and not some other; and to remember that the centre of pressure of a joint is determined entirely by the system of forces acting on the blocks which rest *above* it.

Design of Portion IV.

252. Up to this point the wall's profile has been designed so as to satisfy the first condition of stability alone, para. 243, the fulfilment of this condition providing a wall broad enough at every horizontal joint to bear the vertical load which rests upon it. But below a certain joint, the position of which may be determined by a simple geometrical construction, the profile so designed will be found too narrow to satisfy the second condition of stability, which has reference to the wall's strength, and therefore from this joint downwards the profile must be proportionately widened. The position of this joint may be approximately determined as follows:—

To determine the horizontal joint, dividing Portions III. and IV.—If horizontal measurements be reckoned, as heretofore, from the inner, or wetted, face of the wall as origin, since the line of pressure of the full reservoir falls within the outer half, and that of the empty reservoir within the inner half, of the wall's thickness, the condition of strength

of the two cases of reservoir full and reservoir empty may be symbolically represented by the following expressions :—

For the full reservoir, when the centre of pressure falls within the outer half of the joint, the maximum intensity of pressure

$$= \frac{2T}{\beta} \left\{2 - \frac{3}{\beta}(\beta - \delta)\right\} < \text{ or } = S\left(\frac{1 + \cos\chi}{2}\right) \text{ or K, suppose,}\ldots(16).$$

For the empty reservoir, when the centre of pressure falls within the inner half of the joint, the maximum intensity of pressure

$$= \frac{2T'}{\beta} \left\{2 - \frac{3}{\beta}\delta'\right\} < \text{ or } = S'\left(\frac{1 + \cos\chi}{2}\right) \text{ or K', suppose}\ldots\ldots(17),$$

where T or T′ stands for the total vertical load resting on the horizontal joint whose breadth is β and centre of pressure distant δ or δ', as the case may be, from the inner, or wetted, edge, the batter of the outer face being χ, that of the inner χ', and the corresponding limiting intensity of pressure S, or S′, when the face is vertical.

Hitherto T has been always equal to T′, because the weight of water resting on the inner face of the wall has been left out of account, the batter of that face being so insignificant as to render the obliquity of the fluid pressure on that face with regard to the horizon scarcely worthy of consideration. But if the batter of each face has to be increased in order to obtain the necessary width of profile, and that of the inner face become considerable, the actual direction of the fluid pressure on it must be taken into account, and the vertical component of that pressure, being the weight of the water superincumbent on the inner face, will add materially to the wall's stability, and allow of the masonry being correspondingly diminished in weight. The value of T, therefore, will differ from that of T′ throughout Portion IV. of the wall.

Throughout the whole of Portions II. and III., that is, for the whole portion of the wall's profile designed to satisfy the first condition of stability, para. 243, at each horizontal joint of the full reservoir wall δ is not $> \frac{2}{3}\beta$, and at each one of the empty reservoir wall δ is not $< \frac{1}{3}\beta$, so that, since for this portion of wall $T = T'$, we have the maximum intensity of vertical pressure equal to $\frac{2T}{\beta}$, that is, to twice the mean intensity, and the single condition of strength becomes

$$\frac{2T}{\beta} = K, \ldots\ldots\ldots\ldots\ldots\ldots(18).$$

It will be sufficient to concern ourselves with the outer or dry face only, as the formulæ can be adapted without difficulty to suit the inner face, if necessary.

It has been already pointed out in para. 248 that, at a short distance below the base of Portion I., the surface of the outer face becomes practically an inclined plane, the batter remaining constant for some distance and approximately equal to $\tan^{-1}\left(1{\cdot}03 \times \frac{b_1}{d_1}\right) = \tan^{-1}\left(\frac{1{\cdot}03}{\sqrt{\theta}}\right)$, so that for this portion of the wall the value of K is known. Before, then, the limits proposed by Professor Rankine in the Report referred to in para. 242 can be adopted, the value of S must be determined which will make the expression $S\left(\frac{1 + \cos\chi}{2}\right)$ equal to the weight of 125 cubic feet of the material of which the dam is to be built, for the particular value $\chi = \tan^{-1}\left(\frac{1{\cdot}03}{\sqrt{\theta}}\right)$ of the batter. We have, by equating these quantities,

$$S\left(\frac{1 \times \cos\chi}{2}\right) = 125\ w$$

$$\therefore\ S = \frac{250 \times w}{1 + \cos\chi}$$

in which χ can either be measured from the diagram or calculated numerically as follows:—

$$\cos\chi = \frac{1}{\sqrt{1 + \tan^2\chi}} = \sqrt{\frac{\theta}{\theta + (1{\cdot}03)^2}} = \sqrt{\frac{\theta}{\theta + 1{\cdot}06}} \text{ nearly.}$$

$$\text{Hence } S = 250 \times w \div \left\{1 + \sqrt{\frac{\theta}{\theta + 1{\cdot}06}}\right\}.$$

If $w = 125$ lbs. and $\theta = 2$, the wall being built of brickwork to retain water, weighing 62·5 lbs. per cubic foot, the most common case, then $S = 17{,}284$ lbs. nearly.

If $w = 162{\cdot}5$ lbs. and $\theta = 2{\cdot}6$, the wall being built of granite, likewise to retain water at 62·5 lbs. per cubic foot, then $S = 22{,}007$ lbs. A uniform value of $S = 20{,}000$ lbs. throughout is sometimes adopted for brickwork (*vide* Example II., para. 257 (*d*)).

Now, equations (12) and (15) enable the curve of breadth, corresponding to given increments of height, or rather depth, of the wall, whose profile fulfils the first condition of stability, to be determined; for our present purpose it will be convenient to deal with curves of breadth and length (or rather depth) corresponding to given increments of *weight* of wall. Such curves may be approximately described by laying off from

the load line Ow, as in *Fig.* 97*b*, the breadth and height (or rather depth) of profile corresponding to given lengths of the load line, that is, to given weights of the wall. Thus, for the zones I., II., III.,......the curves of breadth and height (or rather depth) can be at once plotted, as in *Fig.* 97*b*.

For the breadth curve of strength, then, corresponding to given increments of weight, we have the following geometrical construction—

Equation (18) may be written thus—

$$\frac{T}{\beta} = \frac{K}{2} = \frac{nK}{2} \times \frac{1}{n},$$

in which n is any positive integer, and $K =$ the weight of 125 cubic feet of the material of the wall; it evidently expresses the relation T weight units : $\frac{nK}{2}$ weight units :: β length units : n length units, or, $\frac{nK}{2}$ weight units : n length units :: T weight units : β length units.

Now, T is some length measured on the load line Ow, *Fig.* 97*b*; if, therefore, Ow be produced upwards to k, so that $Ok =$ half the weight of $125 \times n$ cubic feet of material $= \frac{125 \times w}{2} \times n$, as measured on the scale of loads, and if a length $kl = n$ feet be then set off at right angles to Ow and on the reverse side of it to that on which the breadth and depth curves are plotted, and the points l and O be joined and lO produced, then will the straight line lOx represent the curve of strength, which fulfils the *second* condition of stability and gives the breadth of profile corresponding to given increments of *weight* of the wall, as measured on the load line Ow. In the example given in *Figs.* 97*a* and 97*b* n is taken $= 20$. The intersection of Ox, therefore, with the breadth curve (which fulfils the *first* condition of stability) already plotted, determines the width xy of the horizontal joint, down to the level of which the profile, as already designed, is both stable and strong enough at every horizontal joint to bear the imposed vertical load. The approximate depth yr of this joint below the water surface may be at once ascertained from the depth (or height) curve already plotted, and the magnitude of the corresponding vertical load $Oy = T$, resting on the joint, by measurement on the load line. These approximations must be checked by comparison with the profile and by calculation of the weight, &c.

Below the critical joint, thus determined, the width of the profile must be increased beyond that given by equations (12) and (15); moreover, the line of resistance of the full reservoir wall will no longer necessarily fall at a distance of two-thirds the wall's thickness measured from the inner, or wetted, face, nor that of the empty reservoir at one-third that thickness, so that the value of δ in equations (16) and (17) must be determined.

To determine the Profile of the Zones.

253. The profile of this portion of the wall is often designed graphically by a process of trial and error, but it will be found more satisfactory to calculate a section by means of the equations following, employing the graphic method to draw in afterwards the lines of resistance.

For the full reservoir, then, we have for the limiting intensity of stress of the outer or dry face the relation, *vide* equation (16),

$$\frac{2\,T}{\beta}\left\{2 - \frac{3}{\beta}(\delta - \beta)\right\} = K$$

$$\text{or } 6\,T\delta - 2\,T\beta = K\beta^2, \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

in which T stands for the total vertical load resting on the horizontal joint whose breadth is β; δ for the distance of its centre of pressure from the inner, or wetted edge, of that joint; K for the limiting intensity of normal resistance $= 125 \times w \times \left(\frac{1 + \cos\chi}{2}\right)$ or $S \times \left(\frac{1 + \cos\chi}{2}\right)$; and χ for the batter of the outer, or dry, face of the wall (para. 242).

Let B, *Fig.* 96*d*, be the breadth of the lowest horizontal joint of the portion of wall already designed, D its depth below the water surface, W the weight of the wall lying on breadth B, and c the distance from the joint's wetted edge of its line of action; then, if s_1 be the batter base of the inner, or wetted, face of a new zone, lying immediately below that one which rests on the joint whose breadth is B, so that B is the top width of the new zone; and if z be the height of the new zone, and s_2 the batter base of its outer face, then will

$$\beta = s_1 + B + s_2, \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

and for the total weight T resting on the new joint whose breadth is β, including the weight of the water superincumbent on the inner face, we have

$$T = (2\,D + z)\frac{w'}{2}s_1 + W'' + \frac{w}{2}(s_1 + 2B + s_2)\,z, \qquad (21).$$

The distance $\bar{c}$ of the line of action of T from the wetted edge of the joint may be determined in the usual way, by equating the moment of T about that edge to the sum of the moments of the several parts making up T about the same edge, thus—

$$6\,T\,\bar{c} = s_1^2\left\{2\,wz + w'(3\,D + z)\right\} + s_1\left\{zw\,(6\,B + 3\,s_2) + 6\,W''\right\}$$
$$+ zw\,(3\,B^2 + 3\,Bs_2 + s_2^2) + 6\,W''c \qquad (22),$$

and the value of δ may be found by substituting for $\bar{c}$ in the relation (as in paras. 216 and 247),

$$\delta - \bar{c} : \tfrac{1}{3}(D + z) : : \tfrac{1}{2}(D + z)^2\,w' : T$$

which reduces to the form

$$6\,T\delta = w'(D + z)^3 + 6\,T\bar{c}$$

so that equation (19) may now be written down in terms of s_1, s_2 and known quantities. By reduction a quadratic for s_1 of the following form may be obtained—

$$Ls_1^2 + Ms_1 + N = O, \qquad (23),$$

in which $L = K - (wz + w'D)$,

$$M = (B + s_2)\{2\,K + w'(2\,D + z) - zw\} - (2\,Bzw + 4\,W),$$

$$N = K\,(B + s_2)^2 + 2\,W\left\{(B + s_2) - 3c\right\} - \left\{zwB^2 + w'(D+z)^3\right\}$$

In which $K = S\left(\frac{1 + \cos\chi}{2}\right)$ where χ is the batter, and S has the definition given in para. 242.

It will be observed that when $z = \frac{1}{w}(K - w'D)$ the co-efficient of s_1^2 vanishes, and that it is positive for all values of z less than this one, and negative for greater values. The height of the zones should, as far as possible, be retained uniform.

In applying the above formula it will be best to assume a value for χ, so that K and s_2 will be known quantities in the equation, and then to determine the corresponding value of s_2; the numerical value of $\beta = s_1 + B + s_2$ will then be known. The value of δ may afterwards be calculated, if necessary, or determined by graphical construction and measurement, and compared with β, and care taken that

$$\delta \text{ is not} > \tfrac{2}{3}\,\beta$$

in order that the first condition of stability may be satisfied for the full reservoir (para. 213).

254. *For the empty reservoir*, by omitting the terms involving w' in equations (21) and (22) it is easy to deduce the results

$$\bar{c}' = \frac{Q}{3R}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

in which

$$Q = 2\,wzs_1^2 + 3\,s_1\left\{wz\,(2\,B + s_2) + 2\,W''\right\} + zw\left\{3\,B^2 + 3\,Bs_2 + s_2^2\right\}$$
$$+ 6\,W''c \text{ and } R = 2\,W'' + wz\,(s_1 + 2\,B + s_2).$$

Care must be taken that $\bar{c}'$ is not $< \frac{\beta}{3}$.

Remarks on Portion IV.

255. In equation (23) it will be observed that s_1 and s_2 vary inversely, and that, in fixing their values, some choice may be exercised within certain limits. For instance, the batter of the outer or dry face might be retained constant throughout Portion IV., that of the inner being varied to suit each zone, as in *Fig.* 97*a*; or, on the other hand, to avoid encroaching on the reservoir, it might be desirable to give as small a batter as possible to the inner face. Equation (23) is obviously better adapted to the case in which the batter of the outer face is assumed and that of the inner made to suit it, because if the value of χ be fixed, then K and s_2 are known, and s_1 can be easily determined, whereas many values for χ might be tried before a value of s_1, suitable for the second arrangement above described, could be obtained. The best method to adopt is, in fact, to retain B, and therefore also c, constant throughout Portion IV., the resulting error, if appreciable, being on the side of safety; the quantity $B - 3c$ will then vanish from co-efficient N in equation (23), since $c = \frac{1}{3}B$. Such a value must be chosen for χ as shall cause $\bar{c}'$ and δ to fall within prescribed limits for the value z_1 of z, being the height of the zone *which comprises the whole of Portion IV.*; K can then be retained constant throughout this Portion (*vide* Example I. following), and values of s_1 corresponding to values of z which are less than z_1 can then be calculated. In this way, after once fixing the value of K and the width of the bottom horizontal joint, the terms in equation (23) involving the symbols z, s_1 and s_2 only will vary. The batter of the outer face will thus be kept constant throughout Portion IV.

Summary of Steps.

256. The following Summary of the several steps in the design of the dam's profile is added for convenience of reference :—

1°. Determine the breadth b_1 of the dam at top, usually an arbitrary quantity, para. 244.

2°. The depth $d_1 = \sqrt{\theta}\, b_1$ of *Portion I.* is then known, para. 246 equation (4).

3°. *Portion II.*—The profile of the outer face can now be designed by equation (11) or (12), the inner face being supposed vertical, para. 248.

4°. *Portion III.*—At a depth of $2{\cdot}1 d_1$ below the dam's top, the inner face ceases to be vertical, and its batter commences, which may be designed by equation (14) or (15), para. 250.

5°. In para. 252 the geometrical construction is described for determining the commencement of *Portion IV.*, the profile of which may be designed by means of equations (23) and (24) of para. 253.

6°. Graphical methods for drawing in the lines of resistance are described in paras. 251(a) and 251(d).

Examples.

257. (a). *Example I.—Figs.* 97a and 97d, *Plate* XXIII., exhibit the rough profile of a dam, 214 feet high, to be built of granite of normal strength, weighing 162·5 lbs. per cubic foot, required to resist the pressure of water, standing flush with its top, the weight of which is taken at 62·5 lbs. per cubic foot. Professor Rankine's limits of safe stress are to be employed throughout, (*vide* paras. 242 and 252).

Fig. 97a illustrates Method I. of para. 251, and Method II. is illustrated in *Fig.* 97d. The former will be seen to be the simpler method.

The following summary of steps is added :—

Portion I.—If H be put = 214 in equation (1), para. 244, then $b_1 = 19$ feet nearly, whence, since $w = 162{\cdot}5$ and $w' = 62{\cdot}5$, we have, by para 246, $d_1 = 30{\cdot}63$ feet.

Portions II. and *III.*—The following values of x and x' for the cor-

responding values of n are given by equation (11) or (12) of para. 248 and equation (14) or (15) of para. 250 :—

$d \div d_1$.	n	k.	FEET.	
			x.	x'.
$1\frac{1}{2}$	$\frac{1}{2}$	1·330	25·27	...
2	1	1·772	33·67	...
$2\frac{1}{2}$	$1\frac{1}{2}$	2·263	42·99	...
3	2	2·772	52 67	1·31
$3\frac{1}{2}$	$2\frac{1}{2}$	3·280	62·32	...
4	3	3·803	72·26	2·96
5	4	4·831	91·79	3·87

The batter of the inner face commences at a depth of 2·4 × 30·63 = 73·5 feet, below the dam's top.

(*b*). *Portion IV.*—The load line and curves of breadth and depth having been plotted, *Fig.* 97*b*, as fully described in paras. 251*a* and 251*d*, if, in the formula $Ok = \frac{1}{2} \times 125 \times w \times n$ of para. 252, n be put = 20, and w = 162·5 lbs., then will Ok = 203,125 lbs., and kl = 20 feet; yr will then be found to measure about 122 feet, and the horizontal joint dividing Portions III. and IV. will, therefore, lie just above the base-joint of Zone IV. It has, for simplicity, been taken as though corresponding with that joint.

In designing the profile of Portion IV. by equation (23) of para. 253, B being put = 74, the method described in para. 255 has been adopted, that is, zone $xS'Ty$, *Fig.* 97*d*, has been first dealt with, and a value of χ, about $42\frac{1}{2}°$, giving K = 19,200 lbs., been found to give satisfactory values for s_1 and s_2 ; the same value has been retained for K in designing the remaining zones, that is, first for Zones V. and VI. taken together, and then for Zone V. taken separately ; the resulting values of s_1 and s_2 for these several systems of zones, taken in combination with the values $x' = 2·96$ and $x = 72·26$ of the base-joint of Zone IV., enable the horizontal measurements to be reduced to the vertical passing through B, and the table given above for Portions II. and III., to be completed, thus—

$d \div d_1$.	n.	MEASUREMENTS IN FEET.			
		s_1.	s_2	x'.	x
5	4	6·27	28	9·26	100·26
6	5	14·5	57	17·5	129·26
7	6	42·5	86	45·5	158·26

(*c*). *Stress-Diagram.*—There is little to be said in explanation of the Stress-Diagram. It will be observed that the figure $Ofhj$O of *Fig.* 97*b* affords a complete diagram of the external loads acting on the dam, of which the portion $Ofhj$O represents the system of fluid pressures. Of these, the horizontal components are measured along the horizontal line Oq, and are given by the parabola Bπ of *Fig.* 97*a*,* and the vertical components are measured along the vertical line qj. These latter are, in fact, the weights of the columns of fluid superincumbent on the several *inclined* surfaces, exposed to fluid pressure, of the zones, so that if s be the batter-base of any one of these zones, z its height, and D the depth of its top below the fluid surface, we have for the weight v of superincumbent fluid

$$v = \frac{w'}{2} \times \left(2\text{D} + z\right) \times s.$$

Thus for Zone V., $v =$ 54219·4 lbs., since $s = 6·3$,
for Zone VI., $v =$ 86674·5 lbs., since $s = 8·21$,
and for Zone VII., $v =$ 348075·0 lbs., since $s = 28$.

These quantities are represented in *Fig.* 97*b* by the straight lines qf', $f'h'$, and $h'j$ respectively, and have been already determined graphically, since the directions of the resultant fluid pressures (being normal to the surfaces) and their horizontal components are known. The graphical and numerical methods serve to check one another.

The Lines of Resistance.—As the method of calculation adopted for the profile of Zones I. to IV. of the dam is based on the assumption that the two lines of resistance, that of the full reservoir wall as well as that of the empty one, lie at a distance of one-third the wall's thickness from the nearer face, the graphical construction of Method I., para. 251,

* Unless the lineal scale be large, these pressures are more safely *calculated*

is only shown for Zones V. to VII. of *Fig.* 97*a*, the construction by Method II. is, however, shown for the entire profile of *Fig.* 97*d*.

(*d*). *Example II.—Fig.* 99, *Plate* XXIV., shows profiles of three dams, superposed on one another for purposes of illustration, each being of the same height, 214 feet, and each designed to resist the pressure of water, as in the previous example.

One of the dams is supposed built of brick, weighing 125 lbs. per cubic foot and capable of resisting a limiting intensity of pressure throughout amounting to 20,000 lbs. per square foot for a vertical face, so that $K = 20{,}000\left(\frac{1 + \cos x}{2}\right)$ in equation (23) of para. 253. This profile is indicated by a thin firm line.

The second profile is to be built of weak but heavy stone, weighing 162·5 lbs. per cubic foot, but no stronger than the brick, (*i.e.*, in this case also $K = 20{,}000\left(\frac{1 + \cos x}{2}\right)$ in Portion IV.) This profile is indicated by a thin broken line.

The third profile is that worked out in Example I., the material being granite of normal strength, and Professor Rankine's limits of strength being employed.

The Dam's Horizontal Trace.

258. With regard to the *plan* of the dam, Professor Rankine makes the following remarks in the Report already alluded to in para. 242, and published in the *Engineer* of 5th January, 1872—

"As regards the effect of giving the wall a curvature in plan, convex towards the reservoir, I look upon this as a desirable and, in many cases, an essential precaution, in order to prevent the wall from being bent by the pressure of the water into a curved shape, concave towards the water, and thus having its outer face brought into a state of tension horizontally, which would probably cause the formation of vertical fissures, and perhaps lead to the destruction of the dam. I consider, however, that calculations of stability which treat the dam as a horizontal arch are so uncertain as to be of doubtful utility; and I would not rely upon them in designing the profile. In fixing the radius of horizontal curvature, I consider that the Engineer should be guided by the form of the gorge in which the dam is to be built, making that radius as short

as may be consistent with convenience in execution, and with making the ends of the dam abut normally against the sound rock at the sides of the gorge."

Col. Wray, R.E., remarks "as regards there being no opening through the dam, it is better to avoid such openings, as likely to induce unequal settlements, and there is seldom any serious difficulty in running a tunnel round the end of the dam in the natural rock to receive the outlet pipe with the necessary arrangements."*

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* Cols. Wray and Seddon's "Instruction in Construction", 3rd Edition, p. 391.

Fig. 90.

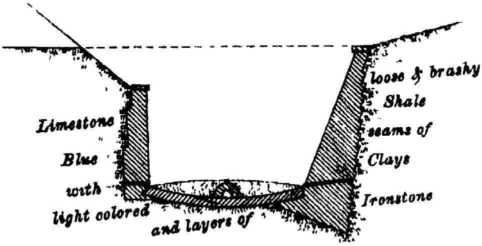

Fig. 91.

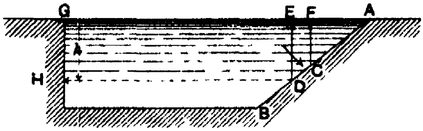

Fig. 92.

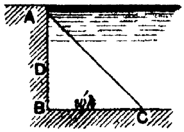

Fig. 93.

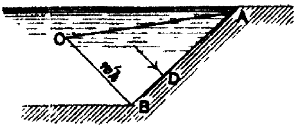

Fig. 94.

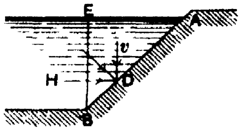

Fig. 95.

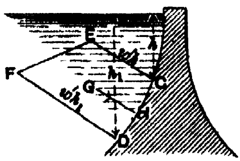

Fig. 96.

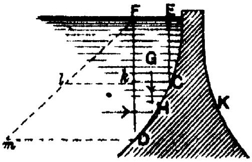

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PLATE XXII.

Fig. 96a.

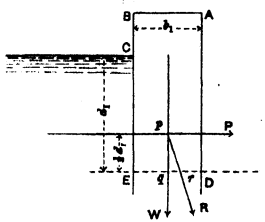


Fig. 96b.

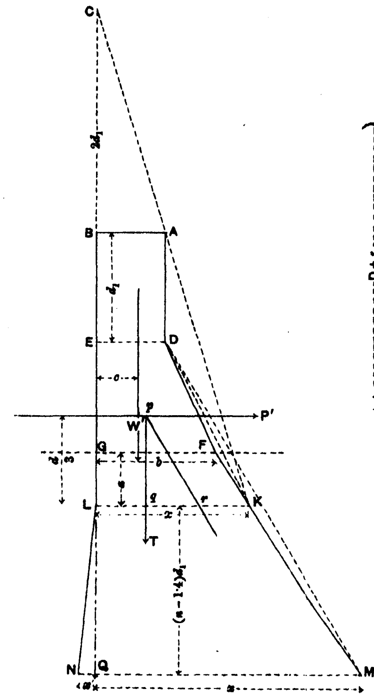


Fig. 96c.

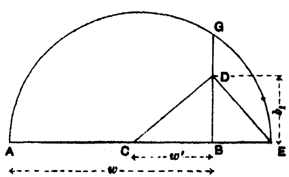
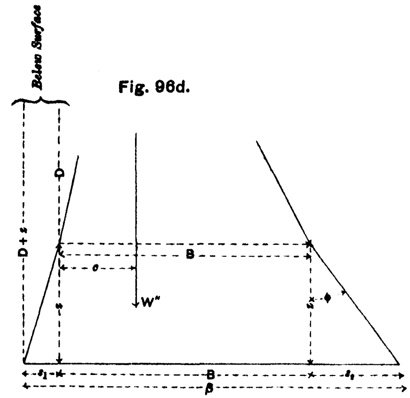


Fig. 96d.



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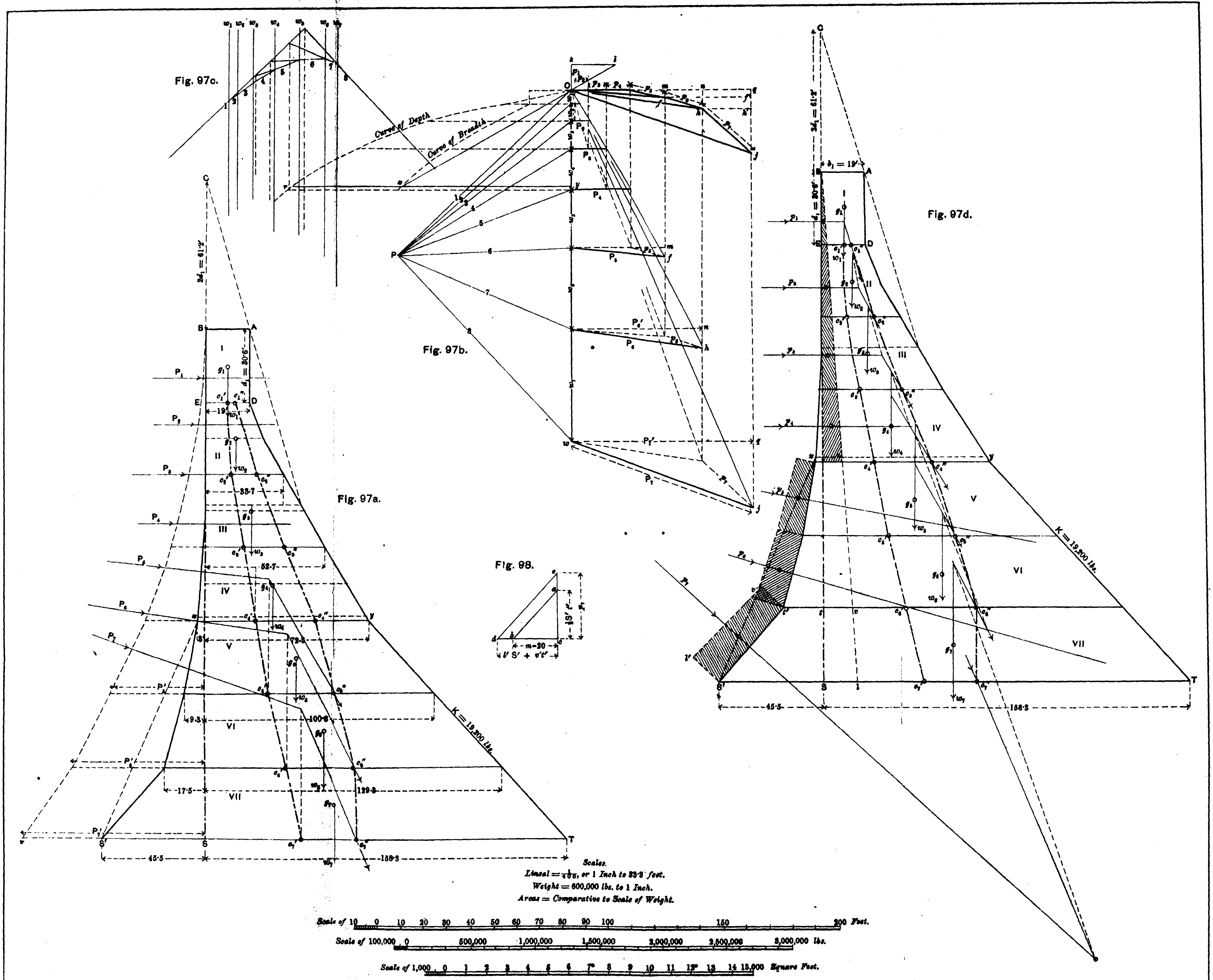

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Fig. 99.

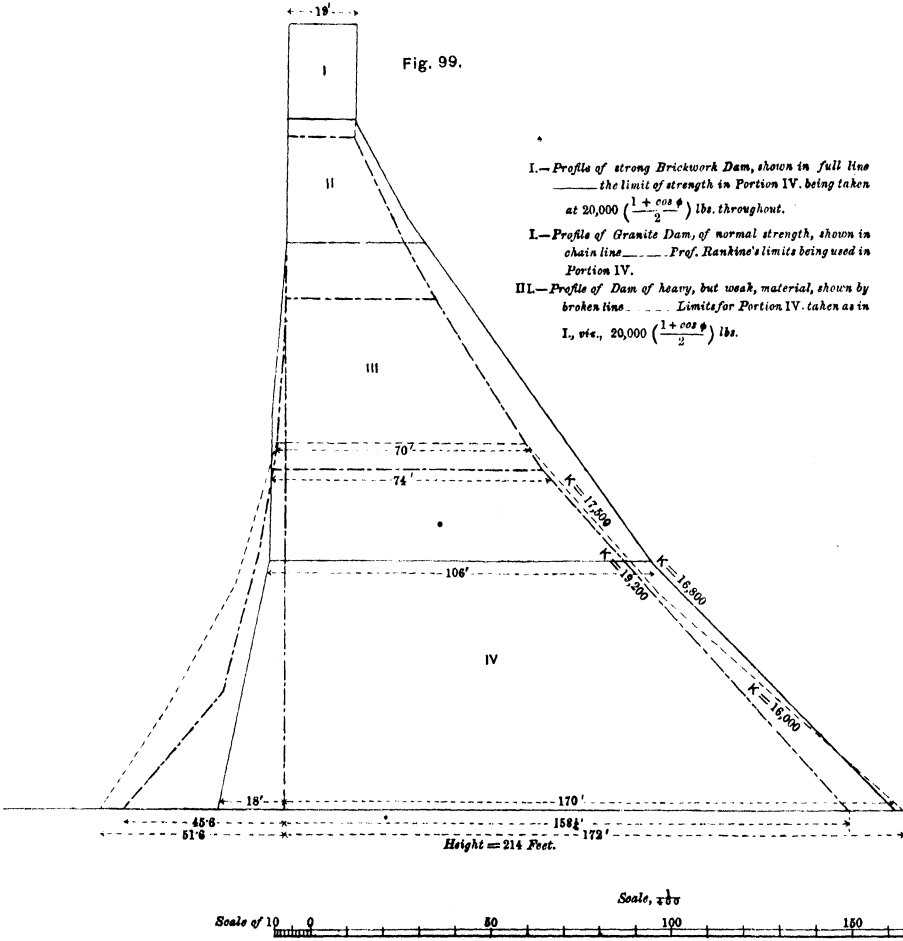


Fig. 100a.

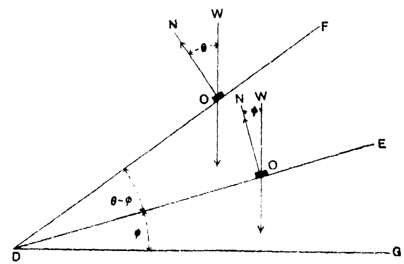
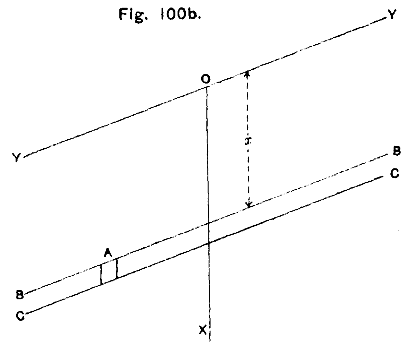


Fig. 100b.



CHAPTER XVI.

THE STATE OF STRESS OF A MASS OF EARTH, CONSIDERED ON ELEMENTARY PRINCIPLES.

259. In the preceding Chapter the pressure of water against a retaining wall was calculated on the supposition that water is a perfect fluid, being frictionless and incompressible. In a similar way, in calculating the pressure of earth against a revetment or retaining wall, *a loose mass of earth may be treated as an imperfect fluid, the particles of which, though incompressible, obey the laws of friction and cohesion.*

260. * "In a mass of earth loaded with its own weight only, the gravitation of the earth causes the vertical pressure, the vertical pressure causes a tendency to spread laterally, and the tendency to spread laterally causes the 'lateral' or conjugate pressure.† Hence the vertical and conjugate pressures stand to each other in the relation of cause and effect or active and passive respectively.

261. "A structure of earth, whether produced by excavation or embankment, preserves its figure at first partly by means of the friction between its grains and partly by means of their mutual cohesion or tenacity, which latter force is considerable in some kinds of earth, such as clay, especially when moist. It is by its tenacity that a bank of earth is enabled to stand with a vertical face, or even an overhanging face for a few feet below its upper edge, whereas friction alone, as will afterwards appear, would make it assume a uniform slope.

262. "But the tenacity of earth is gradually destroyed by the action of air and moisture and of the changes of the weather, so that its friction is the only force which can be relied upon to produce permanent stability. In the present investigation, therefore, the stability of a mass of earth, or of shingle or of gravel, or of any other material consisting of separate grains, will be treated as arising wholly from the

* Rankine's Applied Mechanics, 3rd Edition, pp. 212 and 216.

† A pair of stresses, each acting on a plane, parallel to the direction of the other, are said to be *conjugate* (Rankine's Applied Mechanics, p. 85).

mutual friction of those grains, and not from any adhesion amongst them." The more nearly the mass of earth is reduced by moisture to a fluid condition, the more nearly does the nature of its pressure resemble that of a fluid.

263. In order that a body of any material may rest on a plane of any, whether similar or dissimilar, material by friction alone, the single condition is that the angle between the direction of its pressure on the plane and the normal to the plane shall not exceed the angle of repose of those materials.

Thus, in a mass of loose earth, each particle O on the surface, as on DE, *Fig.* 100*a*, *Plate* XXIV., is from elementary statical considerations stable so long as the direction of its weight, WO, makes an angle WON with the normal ON to the surface DE, not greater than the angle of repose, which is the case so long as the surface slope DE has a uniform inclination not greater than the angle of repose, ϕ. For, were the plane DE to be tilted through the least possible angle FDE, so that the angle θ (or FDG) exceed ϕ by the least amount possible, the particle O would at once slide down the plane, FD, the frictional resistance N tan θ required to balance the tendency of W to slide down the plane, or W sin θ, being greater than the material of the surface is capable of exerting, such limiting value being equal to N tan ϕ only (ϕ being $< \theta$).

264. The same condition which determines the stability along a free surface of earth is also the condition of stability along any plane in the mass. Hence, the mass of earth will be stable if the direction of the resultant pressure between two portions into which it can be divided by any plane in the mass makes an angle with the normal to the plane less than the angle of repose.

265. It will be observed that these conditions leave out of consideration, as has been already pointed out, the forces of cohesion and adhesion, the effects of which act in the same direction as that of friction. The result is therefore on the side of safety.

266. The conditions of stability of a mass of loose earth, with a plane surface of indefinite extent, are fully considered on pp. 212, *et seq.*, of Rankine's Applied Mechanics. It is necessary first to establish the following Three Theorems*:—

* Rankine's Applied Mechanics, 3rd Edition, pp. 126 and 127.

Theorem I.—In an indefinite homogeneous solid, bounded above by a sloping plane, the pressure on any plane parallel to that sloping surface is vertical and of a uniform intensity equal to the weight of the vertical prism which stands on unity of area of the given plane.

For let YOY, *Fig.* 100*b*, represent a vertical section of that sloping surface along its direction of greatest declivity, and OX a vertical plane perpendicular to the plane of vertical section which is represented by that of the paper. Let w be the uniform weight of unity of volume of the substance. Let BB be any plane parallel to, and at a vertical depth x below, the plane YY. If the substance is exposed to no external force except its own weight, the only pressure which any portion of the plane BB can have to sustain is the weight of the material directly above it.

267. Consider the equilibrium of a prismatic particle A of the mass, bounded above by the plane BB, and below by CC, and laterally by two pairs of vertical planes. Then the weight of the prism is equal and opposite to and balanced by the excess of the vertical pressure on its lower face above that on its upper face, and the pressures, parallel to the sloping surface, on its vertical faces, must balance each other independently; therefore they must be of equal mean intensity throughout the whole extent of the layer between the planes, BB, CC. Whence the remaining two Theorems.

Theorem II.—The stress, if any, on any vertical plane is parallel to the sloping surface, and conjugate to the stress on a plane parallel to that surface.

Theorem III.—The state of stress at a given uniform depth below the sloping surface is uniform.

268. The magnitude of the stresses may be measured on elementary statical principles as follows:—

Suppose the earth to the left of the vertical plane AB, *Fig.* 101, *Plate* XXV., to be removed, and a plate, of exactly the same superficial roughness as that of the earth, to be substituted for it. In order to balance the tendency of the mass of loose earth lying behind AB to push the plate back and fall forwards, pressure must be exerted on the plate from the left, and since the nature of earth pressure resembles that of fluid pressure, a resultant effect of the required amount P, applied at a point $\frac{2}{3}$ AB units below A, will be sufficient, as far as the stability of the wedge of earth is concerned, to counteract that tendency. P may, in

fact, be regarded as equivalent to the natural resultant re-action of the earth which originally lay to the left of AB.

269. Now it has been already explained that there is no tendency of the mass to slide down any plane, whose inclination to the horizon is less than the angle of repose ϕ. Consider, then, the tendency to slide down some plane BC, lying between the limiting planes AB and BD, where BD is inclined at ϕ to the horizon.

As in measuring fluid pressure, we may suppose the mass of earth ABC to become solid, and apply the principles of elementary statics to the consideration of its state of equilibrium. Suppose the width of the wedge (perpendicular to the paper) to be unity. It is evidently kept at rest by three forces, viz., its own weight W, acting vertically downwards through its centre of gravity; the conjugate pressure P, acting parallel to the surface, at a point in AB, distant $\frac{2}{3}$ AB below A; and the re-action R of the rough surface BC. These three forces, therefore, must all pass through one and the same point if the wedge is to remain at rest, and this point will be O, since BO $= \frac{1}{3}$ BC, and PO meets AB at $\frac{1}{3}$ AB from B.

270. Since the surface BC is rough, it is evident that the least possible value which P can have is that which enables it to act *in conjunction* with the force of friction, that is, *up* the plane BC. In this case the re-action R, acting through O, is inclined at ϕ° *above* the normal ON, drawn to the surface BC. This value of P, which call P′, just counteracts the natural tendency of the mass of earth to press the plate forward and fall to the left of AB. It is, therefore, of the nature of passive resistance, as far as the mass of earth is concerned, but would be regarded as an active pressure as far as the plate is concerned.

271. Suppose, now, that the pressure of the plate P is increased beyond that *just* necessary to prevent the mass from sliding *down* the plane BC (viz. P′); P then becomes an active force as far as the mass of earth is concerned, and we know from elementary statical considerations that the *greatest possible value* which it can have, provided the state of rest be not disturbed, is that which maintains equilibrium *in opposition* to the force of friction. In this case, the re-action of the plane BC, passing through O, is inclined at an angle ϕ° *below* the normal ON, and we know that if this limiting value of P, which call P″, be exceeded, the wedge ABC will at once commence to slide up-

wards, along the surface BC. The force P″ is of the nature of an active applied force. We know, further, that for all values of P between the limits P′ and P″ the wedge ABC will remain at rest, the resistance R having some inclination less than ϕ to the normal ON, either above or below it, as the case may be, depending on the magnitude of P.

272. Now the sides of the triangles OKR′ and OKR″, *Fig.* 101, are respectively parallel to the groups of the three forces acting at O and keeping the wedge at rest in the two cases; hence, the lengths of the sides of the triangles are respectively proportional to the magnitudes of the forces acting in each case.

Thus, if OK be taken to represent the weight of the same wedge ABC in the two cases, then will OR′ and KR′ represent the resultant pressures R′ and P′ of the wedge against the planes BC and AB respectively at the moment when the wedge is *just about* to slide *down* BC under the action of gravity alone; while the sides OR″ and KR″ will represent the corresponding pressures at the very instant when the wedge is just about to move *up* the same plane, under the action of the impressed force P″ (= R″K) acting in a direction parallel to R″A.

Let the surface plane be inclined at an angle θ to the horizontal. Put $CBD = \chi$, *Fig.* 101. Then, since NOB is, by hypothesis, a right angle, therefore NOK + BOW = a right angle, and therefore $NOK = 90 - BOW = (\phi + \chi)$. But $NOR' = \phi$; therefore $R'OK = \chi$.

Also $R'KO = 90 - \theta$; therefore $KR'O = 90 - (\chi - \theta)$.

Hence, in the first case we have $P' = W \dfrac{KR'}{KO} \propto \triangle ABC \dfrac{\sin \chi}{\cos (\chi - \theta)}$,

and in the second, $P'' = W \dfrac{KR''}{KO} \propto \triangle ABC \dfrac{\sin (\chi + 2\phi)}{\cos (2\phi + \chi - \theta)}$.

273. Now, since the area of the triangle ABC gradually increases as the value of χ diminishes, it is obvious that as χ passes from value zero to value $(90 - \phi)$, *i.e.*, as the plane of cleavage BC moves from coincidence with plane BD to coincidence with AB, the value of P′ passes from value zero, through a maximum value, to value zero again. Similarly, that the value of P″, under like circumstances, passes from a large value when $\chi = 0$ through a minimum value, to a large value again as the area of the triangle ABC approaches value zero, at which point P″ also = zero. These results will be evident from the graphical demonstration given in Chap. XVIII.

274. Now, to apply these considerations to practical cases :—

Suppose AB represent the surface of a retaining wall, the object of which is to keep the mass of earth ABD in position. Obviously the wall must be heavy and strong enough to balance the *greatest possible* pressure that the earth can exert against it. Hence, we must measure a maximum value of P; and since the mass of earth tends to move down the plane BC entirely owing to the action of gravitation, P must be measured *in conjunction* with the force of friction, since the friction of the particles will tend to prevent the motion taking place. In this case, therefore, we must measure *a maximum value of* P′.

275. Suppose, on the other hand, there is a heavy building standing to the left of the plane AB (*i.e.*, on the side opposite to that on which we have supposed the wedge to be situated). The weight of the building will be distributed over the area on which its foundations rest, and will correspondingly increase the intensity of vertical pressure over all horizontal planes beneath that area. Moreover, since the pressure of earth partakes of the nature of fluid pressure, this increased vertical pressure will affect the conjugate lateral pressure all round the building and extending up to the surface, in a somewhat similar manner to that in which a heavy piston pressed vertically on the surface of a liquid affects the lateral pressure against the sides of the vessel all round. The effect of the heavy building standing to the left of the plane AB will, in fact, be to increase the value of P beyond any maximum value it could possibly have under the conditions of the first case considered, that is, above the maximum value of P′; and if we wish to measure the maximum value that P can have under the new conditions, we must remember that, since the force of friction acts as a *preventive* to motion in each case, while in the first case it acted *above* the normal ON to prevent motion *down* the plane, in this case it must act *below* the normal to prevent the increased pressure P causing motion *up* the plane BC; and, whereas, in the first case, we had to provide against the greatest possible passive value that P could have under the given conditions, so as to provide against the likelihood of motion taking place at the *latest* possible moment, in this case we must provide against motion taking place at the *earliest* moment possible, that is, under the action of the *least active force* capable of producing motion at all; so that, while in the first case P′ must have a maximum value, in this case P″ must

have a minimum value; because, when motion does occur, it will take place along the line of least resistance, that is, up the plane of cleavage which can offer the least resistance.

276. Now the positions of these planes of cleavage, and the value of the corresponding maximum and minimum pressure may be determined analytically as follows:—

Let BC, *Fig.* 102, *Plate* XXVI, be any plane of cleavage, AB, as before, representing the vertical plane, the resultant conjugate pressure over which it is required to examine. Let the surface plane be inclined, as before, at θ to the horizon. Draw through B the planes BD, BD″, making the angle of repose ϕ with the horizontal at B, BD being described above, and BD″ below that horizontal plane, and let BD, BD″ meet the surface in the points D and D″ respectively. Through the points B, A, and C, draw the straight lines BG, AF, and CE, each making an angle of $(90 - \theta)$ with BD; and through the same points draw the straight lines BG″, AF″, and CE″, making the same angle $(90 - \theta)$ with BD″. It will be seen at once that BG is inclined at an angle $(\phi + \theta)$ and BG″ at an angle $(\phi - \theta)$ to AB. Also, since BD and BD″ are equally inclined to AB, and AF is inclined to the normal from A on BD at an angle θ, and AF″ to the normal from A on BD″ at the same angle θ, therefore the straight lines AF and AF″ are equal. And since the angle GBG″ $= 2\phi$, and the angle AG″B $= \{(90 - \theta) - (\phi - \theta)\} = 90 - \phi$, because AG″ is inclined to the vertical at $(90 - \theta)$ and BG″ to it at $(\phi - \theta)$, therefore the angle AG″B $= 90 - \phi$, and therefore BG $=$ BG.″

Put CE $= x$, CE″ $= x''$, and let AF $=$ AF″ $= a$, and BG $=$ BG″ $= c$, and put BD $= b$, and BD″ $= b''$. Of these quantities the only variables are x and x''.

Now if the triangles BCE and BCE″ of *Fig.* 102 be compared with the triangles OR′K and OR″K of *Fig.* 101, it will be seen that they are respectively similar; that is, that BCE is similar to the triangle OR′K, and BCE″ to the triangle OR″K. Hence, considering the former pair of triangles, we have—

$$P' = W\frac{KR'}{KO} \propto \triangle ABC\frac{CE}{BE} \propto b\,(a - x)\frac{x}{(b - xK)}, \text{ if } K = \frac{\cos\phi}{\sin(\phi - \theta)}.$$

Hence, differentiating with regard to the variable x, we have

$$\frac{dP'}{dx} \propto \left\{\frac{(b - xK)(a - 2x) + K\,(ax - x^2)}{(b - xK)^2}\right\}$$

and for a maximum value—

$$ab - 2bx + Kx^2 = 0$$

$$\text{or } b\,(a - x) = x\,(b - Kx) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$\therefore$ also $\frac{1}{2}\,(a - x)\,b \cos\theta = \frac{1}{2}\,x\,(b - \text{DE}) \cos\theta$

$\therefore$ $\triangle\,\text{ABC} = \triangle\,\text{BCE}.$

Hence, making the necessary substitution in the relation $\text{P}' \propto \triangle\,\text{ABC}\,\frac{\text{CE}}{\text{BE}}$, we have $\text{P}' \propto x^2$.

Or, taking w = weight of unit volume of the earth,

we have $$\text{P}' = \frac{w}{2}\,x^2 \cos\theta, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In a similar manner, considering the second pair of triangles OR″K and BCE″ of *Figs.* 101 and 102 respectively, it may be shown that—

$$\text{P}'' \propto b\,(x'' - a)\,\frac{x''}{(\text{K}''x'' - b'')}, \text{ where } \text{K}'' = \frac{\cos\phi}{\sin(\phi + \theta)}$$

Differentiating as before, we arrive at the result that, for a minimum value of P″, we must have

$$b\,(a - x'') = x''\,(b'' - \text{K}''\,x''), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

a result exactly similar in form to the one we arrived at when considering P′; whence also it follows that the areas of the triangles ABC and BCE″ must be equal; and also that, for a minimum value,

$$\text{P}'' = \frac{w}{2}\,x''^2 \cos\theta, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

277. Having thus reduced the expressions for $\frac{d\text{P}'}{dx}$ and $\frac{d\text{P}''}{dx''}$ to the same form, we may write x for x'' and b for b'' and deal with them together. Solving the equation we find $x = \frac{b}{\text{K}} \pm \sqrt{\frac{b}{\text{K}}\left(\frac{b}{\text{K}} - a\right)}$.

Now $\frac{b}{\text{K}}$ is in each case $= c$, being in the case of P′ equal to $\text{DB} \div \frac{\text{DE}}{\text{CE}}$ or BG, and in the case of P″ equal to $\text{D}''\text{B} \div \frac{\text{D}''\text{E}''}{\text{CE}''}$ or BG″, whence generally,

$$x = c \pm \sqrt{c\,(c - a)}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

In which expression $(c - a)$ is positive, since BG is always > AF and BG″ > AF″.

If $\frac{d\text{P}}{dx}$ be again differentiated we shall have $\frac{d^2\text{P}}{dx^2} \propto \left(\frac{1}{x - c}\right)^2$, so that the negative value of the radical will give a maximum, and the positive a minimum, value.

Substituting these values of x in equations (2) and (4) we have

$$\left.\begin{aligned} P' = \frac{w}{2}\cos\theta\,\{c - \sqrt{c\,(c-a)}\}^2 = \frac{w}{2}\cos\theta\,\{2c^2 - ca - \\ 2c\sqrt{c\,(c-a)}\},\ldots\ldots\ldots\ldots \end{aligned}\right\} (6).$$

$$\left.\begin{aligned} P'' = \frac{w}{2}\cos\theta\,\{c + \sqrt{c\,(c-a)}\}^2 = \frac{w}{2}\cos\theta\,\{2c^2 - ca + \\ 2c\sqrt{c\,(c-a)}\},\ldots\ldots\ldots\ldots \end{aligned}\right\} (7).$$

Putting AB $= h$, and remembering that now $c = h\,\frac{\cos\theta}{\cos\phi}$ and $\frac{a}{h} = \frac{AF}{AB} = \frac{\sin ABF}{\sin AFB} = \frac{\cos\phi}{\cos\theta}\,a = h\,\frac{\cos\phi}{\cos\theta}$, we have, substituting these values in equations (6) and (7) and reducing

$$P' = \frac{wh^2}{2}\cos\theta\left\{\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}\right\}, \ldots\ldots\ldots\ (8).$$

$$P'' = \frac{wh^2}{2}\cos\theta\left\{\frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}\right\}, \ldots\ldots\ldots\ (9).$$

Or, if p' and p'' be the corresponding intensities of pressure at depth h, we have, remembering that the intensity of pressure increases with the depth, it being of the nature of fluid pressure,

$$p' = wh\cos\theta\left\{\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}\right\}, \ldots\ldots\ldots (10).$$

$$p'' = wh\cos\theta\left\{\frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}\right\}, \ldots\ldots\ldots (11),$$

so that the proportion in which the maximum intensity of conjugate pressure consistent with stability of the mass of earth exceeds the minimum is given by the expression

$$\frac{p''}{p'} = \left\{\frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}\right\}^2.$$

These values are the same as those given on pages 214 and 216 of Rankine's Applied Mechanics, which are arrived at by the application of his *Ellipse of Stress.*

278. But these conjugate pressures are, by the primary definition of them given in para. 260, proportional to the vertical pressures which cause them. Hence, if w'' be the intensity of vertical pressure over unity of area of a plane, parallel to the surface, producing the conjugate pressure p'', and w' that causing p', we have

$$\frac{w''}{w'} = \left\{\frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}\right\}^2 \ldots\ldots\ldots\ldots (12).$$

279. The practical application of this relation is, of course, to the case in which the surface of the ground is horizontal ($\theta = 0$), and the intensity of increased vertical pressure w'' that due to the weight of a building, the plane of whose foundation area is parallel to that of the ground surface, *i.e.*, horizontal also.

If A be the area of the foundation of the building, and h the depth of its plane below the ground surface, and w the weight of a unit volume of earth, then whA will measure the weight of the earth displaced by the building; and if the weight of the building be W, and it be uniformly distributed over the area of its foundation, then will $\frac{W}{A} = w''$ measure the intensity of vertical pressure, due to that weight, and the *limit* of that intensity of pressure is therefore fixed by the relation, putting $\theta = 0$ and $w' = wh$ in equation (12),

$$w'' = wh\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2 \quad \text{.........................(13).}$$

This is *the greatest intensity of vertical pressure, consistent with stability, of a building, founded on a horizontal stratum of earth at the depth h, the angle of repose of the earth being* ϕ (Rankine's Applied Mechanics, p. 220).

280. If, however, the pressure of the building be not uniformly distributed over its base, its *greatest* intensity must not exceed that given by equation (13), and its *least* intensity must not fall short of wh.

281. This condition enables the *greatest inequality of distribution* of the pressure of a building, which is consistent with the stability of a given kind of earth, to be determined.

The most useful and frequent example of this case is that in which the base is rectangular, and the intensity of the pressure increases at a uniform rate from one edge to the opposite edge of the rectangle, being, therefore, of the nature of a *uniformly varying stress*. *Vide* paras. 178 and 182.

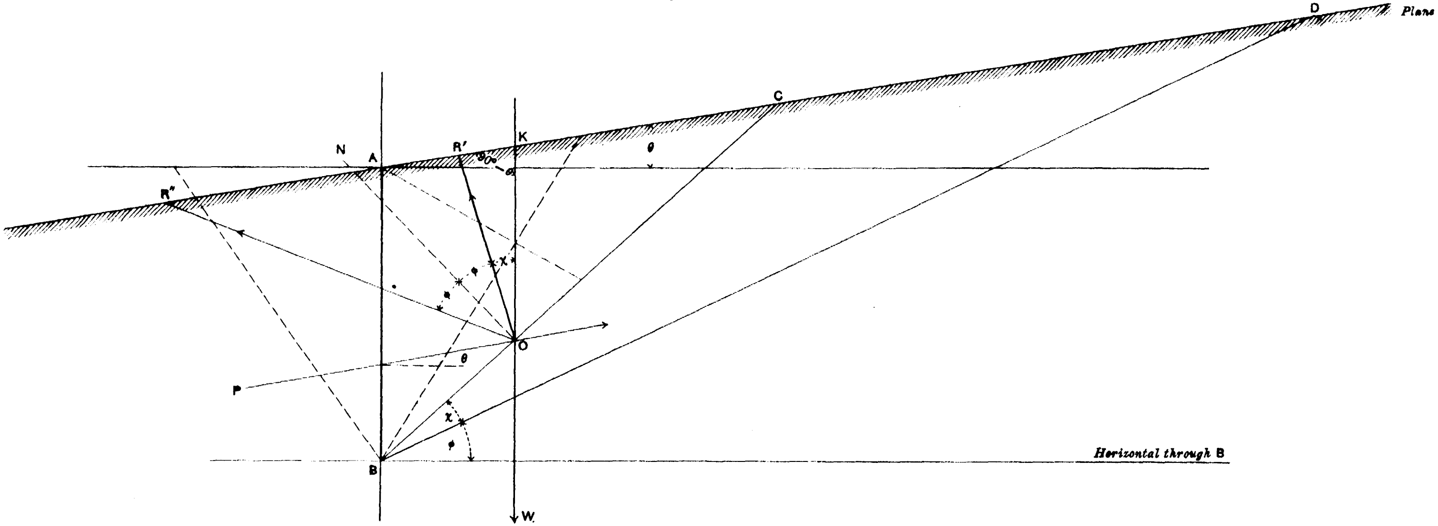
In this case Y of para. 182 $= wh\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2$ and $y = wh$, and we have

$$n = \frac{Y + y}{2} = wh\,\frac{1 + \sin\phi}{(1 - \sin\phi)^2}$$

and we may therefore determine, by the equations of para. 182, the utmost deviation permissible of the centre of resultant pressure on the base of the superstructure from its centre of figure.

282. It must be remembered that in the above investigations the

Fig. 101



Litho. T. C. Press, Roorkee.

state of stress of a mass of loose earth, the surface slope of which is uniform, has alone been dealt with; and further, that the cohesion of the earth, a most important element of stability, has been left out of consideration altogether, thus rendering the results mere approximations on the side of safety.

283. The resultant pressure of masses of earth whose surfaces are irregular against retaining walls, including cases of surcharges, ramparts, and buildings situated near the retaining wall, are best treated by the General Graphic Method, which is explained in Chapter XIX.

284. A most important application of the results of the preceding investigations is to the question of the necessary depth of well foundations in yielding soil. This question has been thoroughly dealt with by Major (now Lieut.-Colonel) Cunningham, R.E., and the results of his investigations will be found in the succeeding Chapter and in Appendix D.

CHAPTER XVII.

ON THE DEPTH OF WELL-FOUNDATIONS.

285. The present Chapter consists of an investigation of the necessary depth of Well-Foundations with vertical sides in yielding soil, *sufficient only to prevent subsidence.* The question of the depth necessary to secure Stability of Rotation under exposure to high winds, current pressure, and other horizontal forces, is fully discussed in Colonel Cunningham's pamphlet on Well-Foundations, which is partly reprinted in Appendix D.

286. Before proceeding to the investigation, however, it will be instructive to remind the reader of the difference in effect, as regards the conjugate earth pressure, between wells, properly hearted with concrete, forming solid masonry structures, the bases of which distribute their pressure more or less evenly over the soil, and the same wells, unhearted or hollow, *Fig.* 103.

Before hearting, the wells are hollow structures, or shells, fitting tightly into vertical holes cut for them in the soil, their lower extremities being wedge-shaped, with cutting edge. After hearting they form solid pillars, which are called on to bear the weight of a heavy superstructure, and which are immersed, or practically float, in a very imperfect fluid, which they not only displace but compress vertically. It will be thus seen that while little, if any, direct support to the vertical load can be afforded at the base of an unhearted well and consequently the lateral or conjugate earth pressure remains to the same extent unaffected, hearted wells, on the other hand, are capable of evoking the utmost intensity of vertical resistance, and so also of conjugate lateral pressure, of which the soil is capable, because the vertical load of the whole superstructure, less the resultant friction against the vertical sides, may be distributed more or less evenly over the base-area. In the case of unhearted wells, therefore, the least possible intensity of

conjugate earth pressure or $p' = wh\left(\frac{1 - \sin\phi}{1 + \sin\phi}\right)$ must be provided for; in the case of hearted wells, the greatest possible, or $p'' = wh\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)$.

It will only be necessary to examine the case of vertical wells, properly hearted with concrete, forming the foundation of a heavy superstructure, the weight of which is thus transmitted fairly evenly over the soil on which it rests. The following paras., 287 to 297, are to a large extent taken from Colonel A. Cunningham's Paper on Well-Foundations, published in the Roorkee "Professional Papers on Indian Engineering," Second Series, Vol. IV., No. CLIII.

287. Such being the case, it seems obvious that the sole SUPPORTING FORCES which prevent subsidence of a Well-Foundation with vertical sides are—

1°. Vertical upward Re-action (R) of the subsoil against the base,—including in this of course fluid-pressure, if water has access to the base (as in a pervious subsoil).

2°. Vertical Friction (F) of the subsoil against the masonry.

It is clear that if—

R = Maximum vertical Re-action developable in subsoil against base of one Well.

F = Maximum vertical Friction developable in subsoil against sides of one Well.

W = Weight of one Well with its Superstructure and Live Load, then the Well must sink so long as $R + F < W$, and will sink no further when $R + F =$ or $> W$, and that if R, F, W be expressed in terms of the depth, then the solution of the equation

$$R + F = W, \quad \text{.........................} \quad (1)$$

will give the value of the least necessary depth, to prevent actual subsidence, and on the other hand for a given depth the ratio

$$(R + F) \div W = \text{Factor of Safety} \quad \text{...............} \quad (1\text{A}),$$

or shows the excess of the Supporting Forces over the Weight of Well and Superstructure.

288. The internal subsoil-pressure at any depth below the bed is the sum of the pressures due to three causes—

1°. Atmospheric pressure.

2°. Pressure due to depth of water in stream.

3°. Internal pressure in the subsoil due to its own weight.

The first of these may be neglected for all practical purposes, and it is wise to neglect the second also, as though increased depth of water undoubtedly increases the subsoil internal pressure *cæteris paribus* in proportion to the depth of water, still this increase cannot be depended on, and it is more than probable that certain physical causes, such as increased permeation of the soil, tend (with increased head of water) to loosen the coherence of a pervious subsoil—which is the case in hand—and *pro tanto* diminish its power of resisting pressure and of yielding friction.

It will, therefore, be assumed that the internal subsoil pressure is that due to the third cause, *i.e.*, to its own weight only. In order to obtain a definite solution, moreover, it will be supposed that *the subsoil is homogeneous*, that the *level of the bed is maintained uniform*, and that the *quality of the subsoil is invariable*, that is to say, that its power of resisting pressure and yielding friction are unaffected by any change in the depth of water.

These limitations make the quantities ϕ, μ, w'', which express the physical properties of the subsoil in the following investigation, constant.

289. Let A = area of base of Well *in sq. ft.*,
C = circumference of Well *in ft.*,
w = heaviness of masonry *in lbs. per c. ft.*,
h = depth of base of Well below surface of soil *in ft.*,
w = weight of masonry of one Well above surface of soil with weight of superstructure and Live Load *in lbs.*

Then, since the Well has vertical sides—

$\mathrm{A}h$ = volume of masonry below surface of soil *in c. ft.*,
$w\mathrm{A}h$ = weight of masonry below surface of soil *in lbs.*,
$\mathrm{C}h$ = surface of masonry in contact with soil *in sq. ft.*,

$$\therefore \mathrm{W} = \mathrm{w} + w\mathrm{A}h, \qquad \text{..........(2).}$$

Let ϕ = angle of repose of subsoil,
μ = co-efficient of friction between subsoil and masonry,
= tangent of angle of repose of the soil on the masonry,
w'' = heaviness of subsoil *in lbs. per c. ft.*,
x = any depth below surface of soil *in ft.*

Then by para. 277 (or Rankine's Applied Mechanics, Art. 199), the Well being properly hearted:—

$$w''x\,\frac{1+\sin\phi}{1-\sin\phi} = \left\{\begin{array}{l}\text{maximum subsoil } \textit{horizontal} \text{ pressure-intensity}\\ \quad\text{at depth } x,\end{array}\right.$$

$$= \left\{\begin{array}{l}\text{maximum subsoil normal pressure upon the}\\ \quad\text{masonry at depth } x,\end{array}\right.$$

$$\therefore\ \mu w''x\,\frac{1+\sin\phi}{1-\sin\phi} = \left\{\begin{array}{l}\text{maximum vertical friction-intensity between}\\ \quad\text{masonry and subsoil at depth } x.\end{array}\right.$$

Thus the maximum vertical friction-intensity developable at any depth varies as the depth, or is a '*uniformly-varying*' *stress*, (para. 178.)

Hence by the known laws of such stress,—

$$\left.\begin{array}{l}\text{Mean intensity of max. ver-}\\ \quad\text{tical friction over a vertical}\\ \quad\text{plane of depth } h,\end{array}\right\} = \tfrac{1}{2}\text{ of}\left\{\begin{array}{l}\text{intensity of the same at}\\ \quad\text{depth } h,\end{array}\right.$$

$$= \tfrac{1}{2}\,\mu w''h\,\frac{1+\sin\phi}{1-\sin\phi},$$

$$\left.\begin{array}{l}\therefore\ \text{Total vertical Friction against}\\ \quad\text{sides of Well of depth } h,\end{array}\right\} = \tfrac{1}{2}\,\mu w''h\,\frac{1+\sin\phi}{1-\sin\phi}\,Ch,$$

$$\therefore\ F = \tfrac{1}{2}\,\mu w''\,\frac{1+\sin\phi}{1-\sin\phi}\,Ch^2, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also by para. 279, or Rankine's Applied Mechanics, Art. 199,

$$w''x\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2 = \left\{\begin{array}{l}\text{max. subsoil vertical (upward) pressure-}\\ \quad\text{intensity at depth } x,\end{array}\right.$$

$$\left.\begin{array}{l}\therefore\ \text{Vertical Re-action of soil}\\ \quad\text{against base,}\end{array}\right\}\text{or } R = w''h\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2 A, \quad\ldots\ldots\ldots(4).$$

290. It will be observed that R varies as the depth, and that F varies as the square of the depth, so that below a certain depth, which may be found by equating equations (3) and (4), F increases much more rapidly with the depth than R; also that in very yielding soils for which ϕ is of course small, $\frac{1+\sin\phi}{1-\sin\phi}$ is not much > 1, so that if the depth h is *large*, as is necessarily the case with slender Wells, F is a much larger quantity than R, which proves what is *already practically known* (*see* Art. 13 of Paper LXXXIII. on Well-Foundations, which is partly reprinted as Appendix D of this Volume), that—

"The Vertical Friction is the principal Supporting Force in case of slender Well Foundations in water-logged sand."

Hence substituting from (2), (3), (4) into (1), the resulting equation for finding h proves to be an ordinary quadratic.

$$\tfrac{1}{2}\mu w''\frac{1+\sin\phi}{1-\sin\phi}Ch^2 + w''\cdot\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2 Ah = \text{w} + w\cdot Ah\ldots\ldots\ldots(5).$$

Writing
$$\frac{1-\sin\phi}{1+\sin\phi} = k',\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$h^2 + \frac{2}{\mu}\left(\frac{1}{k'} - k'\frac{w}{w''}\right)\frac{A}{C}h = \frac{2k'}{\mu w''}\cdot\frac{\text{w}}{C},\ldots\ldots\ldots\ldots(7),$$

the *general* equation from which h is to be found.

291. *Case of cylindric Well.*—If the submerged portion of the Well be a cylinder, of radius r (in feet), then $A = \pi r^2$, $C = 2\pi r$, and Eq. (7) becomes—

$$h^2 + \frac{1}{\mu}\left(\frac{1}{k'} - k'\frac{w}{w''}\right)rh = \frac{k'}{\mu w''}\cdot\frac{\text{w}}{\pi r},\ldots\ldots\ldots\ldots(7\text{A}).$$

292. *Case of Quicksand.*—This is by far the most important case in practice—being that of many large Indian rivers. Adopting the conclusion (*v. supra*) that R is a very small quantity compared with F, and may as an approximation be neglected, the fundamental equation (1) becomes

$$F = W,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and Eq. (7), (7A) become

General case— $$h^2 - \frac{2k'}{\mu}\cdot\frac{w}{w''}\cdot\frac{A}{C}h = \frac{2k'}{\mu w''}\cdot\frac{\text{W}}{C},\ldots\ldots\ldots\ldots\ldots(9).$$

Cylindric Well— $$h^2 - \frac{k'}{\mu}\cdot\frac{w}{w''}\cdot rh = \frac{k'}{\mu w''}\cdot\frac{\text{w}}{\pi r},\ldots\ldots\ldots\ldots\ldots(9\text{A}).$$

The error made in the value of h resulting from Eq. (9), (9A), due to neglecting R, is on the side of safety.

293. *Example.*—In the Kali Nadi Viaduct, a weight of 414·4 tons rests on each well above level of bed; the wells are 12 feet cylinders; the subsoil is said (by the Resident Engineers) to be of such a nature that

$$\phi = 15°,\ \mu = \tfrac{1}{3},\ w'' = 100 \text{ lbs. per c. ft.}$$

Find least depth h necessary to prevent actual subsidence.

Solution. Here $k' = \dfrac{1-\sin\phi}{1+\sin\phi} = \cdot588$, $\dfrac{1}{k'} = 1\cdot7$ (Rankine's Applied Mechanics, Art. 201), $\dfrac{1}{\mu} = 3$, $r = 6'$, $w = 120$ lbs. per c. ft., $\text{w} = 414\cdot4 \times 2{,}240$ lbs.

Hence Eq. (7A) becomes

$$h^2 + 3\times\left(1\cdot7 - \cdot588\times\frac{120}{100}\right)\times 6h = \frac{3\times\cdot588}{100}\times\frac{414\cdot4\times2240}{\frac{22}{7}\times 6}$$

whence $h^2 + 18h = 868\cdot3$, *nearly.*

$$\therefore h = -9 \pm \sqrt{868\cdot3 + 81} = -9 \pm \sqrt{949\cdot3} = -9 \pm 30\cdot8$$

$$\therefore h = 21\cdot8, \text{ nearly.}$$

This is the very least depth at which under all the hypotheses the Well would just *cease to sink*—in fact it might be said to be *floating* at this depth, its weight being just balanced by the Supporting Forces.

294. The Well is, of course, sunk hollow, the soil being excavated to receive it, and the process of hearting accomplished after it has been sunk to the required depth. It is known that in many cases in the process of sinking, the subsoil is *completely removed* from underneath the base of the Well, at which time the Vertical Re-action (R) of the subsoil can only be the hydrostatic pressure due to the head of water within the well.

295. With regard to the values given to the constants ϕ, μ, and w'' in the above example, Colonel Cunningham in the paper alluded to makes the following remark :—

"It seems doubtful if the values $\phi = 15°$, $\mu = \frac{1}{3}$, can be depended on, as they are apparently the figures for *masonry in contact with moist clay* of Art. 110 of Rankine's Civil Engineering. The value $w'' = 100$ lbs. per cubic foot also seems very large for quicksand."

He, therefore, proposes a method, similar to the following, for practically measuring the value of the Total Vertical Friction (F) developed.

296. Supposing the Well to have vertical sides both externally and internally, and in addition to previous notation, supposing

a = Area of hollow at base of Well in square feet,

w' = heaviness of water = $62\frac{1}{2}$ lbs. per cubic foot,

h'' = head of water inside* Well above the base.

Then $R = w' (A - \mathrm{a}) h''$, ... (10).

$W = \mathrm{w} + w (A - \mathrm{a}) h$, ... (11).

In general during the progress of sinking, $R + F < W$, but in the state *just before actual subsidence occurs* (the Well being unhearted),

$$R + F = W,$$

whence,
$$\tfrac{1}{2} \mu w'' \cdot \frac{1 - \sin\phi}{1 + \sin\phi} \cdot Ch^2 + w' \cdot (A - \mathrm{a}) h'' = W,$$

$$\left.\begin{aligned} \therefore \mu \cdot \frac{1 - \sin\phi}{1 + \sin\phi} &= 2 \cdot \frac{W - w' (A - \mathrm{a}) h''}{w'' Ch^2} \\ &= 2 \cdot \frac{\mathrm{w} + (wh - w'h'') (A - \mathrm{a})}{w'' Ch^2} \end{aligned}\right\} \text{........} (12),$$

in which equation w, w', w'', C, A, a are all constant, and are known quantities, and w, h'', h are quantities which vary during the progress of sinking, and are the *quantities to be observed*, whilst μ, ϕ are the quantities sought.

* This is not necessarily the same as the head outside the Well

297. It may be objected to the calculation here proposed that there is *only one equation* (12) to determine *two* quantities μ, ϕ. The result of calculation from Eq. (12) will of course give *only the value of the quantity* $\mu \cdot \frac{1 \pm \sin\phi}{1 \mp \sin\phi}$, (not of either μ or ϕ separately,) whereas the use of the general equation (7) or (7A) above requires a knowledge of both μ and ϕ.

It must be observed, however, that the quantity found $\left(\mu \cdot \frac{1 \pm \sin\phi}{1 \mp \sin\phi} = \frac{\mu}{k'}\right)$ is sufficient for the determination of the Total Vertical Friction (F) by Eq. (3), and that in the very case of most practical importance—that of Well-Foundations *in quicksand*—this has been explained above (para. 292) to be by far the most important element of vertical Re-action, and is all that is wanted for solution of the special equations (9) or (9A) for quicksand, so that the application of this Method, though insufficient *in general* (*i. e.*, for any soil), would be of great importance for quicksand.

If a sufficient number of observations were obtained of the quantities w, h'', corresponding to different depths (h), the correctness of the Theory could also be tested. Colonel Cunningham adds :—

" Great care would be necessary to use only such observations of w, h'', h as were really suitable for the object intended. It is absolutely necessary that—

1°. There should be no subsoil below the base of the Well at time of observation.
2°. The observations be recorded just at the time that actual subsidence is beginning.
3°. The *quality and state of aggregation* of the subsoil at well-base be also simultaneously recorded.
4°. The *mean level* of actual bed of the stream near the masonry be also simultaneously recorded.

" Unless these precautions be carefully attended to, the resulting values of $\mu \div k'$ (for the same soil of course) would be so discordant as to be obviously inaccurate, and it would be hopeless to attempt to reconcile them.

" But it seems probable that the observations do not admit of accurate determination, in consequence of the difficulty of ascertaining whether the necessary conditions were approximately fulfilled, so that some discordance must be expected in the results,—as is always the

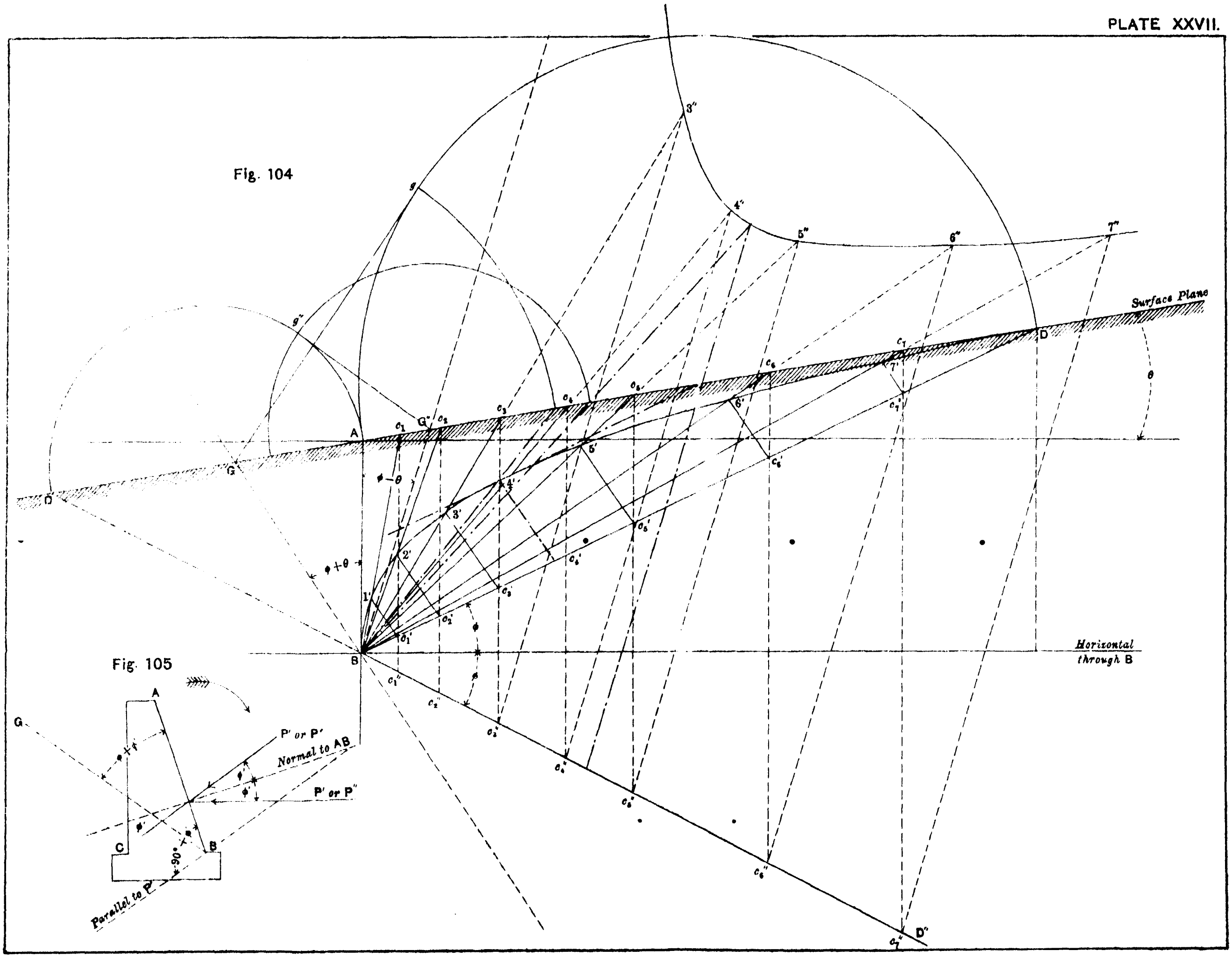


Fig. 104
Fig. 105
Surface Plane
Horizontal through B
Normal to AB
P' or P''
Parallel to P'

case in the determination of all physical constants, and it would probably be a tedious work to reconcile them.

"The importance of the result aimed at seems, however, to make it advisable to attempt to determine this quantity ($\mu \div k'$) for water-logged sand, such as is common in Indian rivers. If this constant were known, the Problem of STABILITY OF ROTATION of Well-Foundations treated of in (Colonel Cunningham's) Paper, No. LXXXIII. of the Roorkee 'Professional Papers,' (and partly reprinted in Appendix D of this Volume) "could also be *completely solved*, so that the solution of the Problem, of necessary Depth of Well-Foundations in quicksand *in all its aspects* could then be effected with some chance of approximation."

298. It will be interesting to trace the relative change of value obtaining between a variable depth h and the corresponding values of the resistance referred to in para. 289, that is, between the values of F and h on the one hand and R and h on the other.

If h be measured on some lineal scale vertically, and the corresponding values of F and R on some scale of loads horizontally, it will be evident from equations (3) and (4) of para. 289 that the extremities of successive values of F will trace a parabola, whose apex is at point $h = 0$ and axis coincident with the surface; also that the direction of measurement of h is that of the tangent to the parabola at the apex. It is also evident that the extremities of successive values of R trace a straight line, which is inclined to the vertical and passes through the point $h = 0$. The straight line and parabola evidently intersect at the depth given by equating the values of R and F in equations (3) and (4), that is, at the point where $h = \frac{2}{\mu}\left(\frac{1 \pm \sin\phi}{1 \mp \sin\phi}\right)\frac{A}{C}$, the upper sign being used in the case of solid, and the lower in that of hollow, wells. This relation is graphically represented in *Fig.* 103.

Thus, when sinking an unhearted well, the resistance is at first almost entirely direct and due to the pressure on the annular base. At the depth given by the lower signs of the above equation the lateral and basic resistances are equal, and at greater depths the former resistances rapidly increase.

CHAPTER XVIII.

GRAPHICAL DETERMINATION OF THE MAXIMUM AND MINIMUM CONJUGATE PRESSURES AND THEIR PLANES OF CLEAVAGE.*

299. In the relations (para. 272 and *Fig.* 101) $P' \propto \triangle ABC \frac{\sin \chi}{\cos(\chi - \theta)}$ and $P'' \propto \triangle ABC \frac{\sin(\chi + 2\phi)}{\cos(2\phi + \chi - \theta)}$, it is evident that as χ increases the area of the triangle ABC diminishes, and *vice versâ*, so that when $\chi = O$, $P' = O$, and P'' has some numerical value. As χ increases, the value of P' increases up to a certain point and then diminishes, while that of P'' gradually diminishes and then increases, until, when $(2\phi + \chi - \theta) = 90°$; that is, when $\chi = (90° + \theta - 2\phi)$; that is, when the plane of cleavage corresponds with BG'', *Figs.* 102 and 104, P'' has an infinite value

300. The several values of the resultant pressures P' acting parallel to the surface plane and at a distance $= \frac{2}{3}$ AB below A, and corresponding to the several planes of cleavage Bc_1, Bc_2, Bc_3 ...... Bc_7, BD, *Fig.* 104, *Plate* XXVII., may be graphically determined as follows :—

Let AB, as before, represent a vertical plane dividing the mass, and through B, as before, let the plane BD be described making an angle equal to $\phi°$, the angle of repose of the earth, with and *above* the horizontal drawn through B, and let BD'' be described through the same point B making an angle ϕ with and *below* the same horizontal.

Through the points c_1, c_2, c_3 ......, &c., in which the planes of cleavage meet the surface, draw the verticals $c_1c_1'c_1''$, $c_2c_2'c_2''$, $c_3c_3'c_3''$, &c., to meet the planes of repose BD and BD'' in the points c_1' and c_1'', c_2' and c_2'', c_3' and c_3'', &c., respectively.

Then it is evident, since the triangles ABc_1, ABc_2, ABc_3, &c., have the vertex B common that their areas are proportional to the lengths

* This method is taken from Chalmer's "*Graphical Determination of Forces in Engineering Structures*," Chap VII.

of their bases, Ac_1, Ac_2, Ac_3, &c.; that is, to the lengths Bc_1', Bc_2', Bc_3', &c.; or the lengths Bc_1'', Bc_2'', Bc_3'', &c.

If, now, through the points c_1', c_2', c_3', &c., we draw straight lines inclined at $(90 - \theta)$ to BD, that is, parallel to BG, to meet the corresponding planes of cleavage in the points 1′, 2′, 3′, &c., then will the series of triangles $Bc_1'1'$, $Bc_2'2'$, $Bc_3'3'$, &c., have their sides mutually inclined to one another at the same angles as are the forces *taken in order*, which act on the corresponding wedges of earth ABc_1, ABc_2, ABc_3, &c., when motion is *just about* to take place *down* the corresponding plane of cleavage; and the triangle OKR′, *Fig.* 101, *Plate* XXV., may in fact be regarded as representative of any one of them, after revolution, in the direction of the hands of a watch, through an angle $(90° - \phi)$.

So that, if the weight of any one of the wedges, as ABc_1, be numerically calculated, and the quantity so obtained be compared with the representative straight line, as Bc_1', the *scale* of all the other representative quantities will be at once known; and, therefore, also the magnitudes of the several conjugate pressures P′, corresponding to the several planes of cleavage when motion is *just about* to take place *down* them and acting parallel to the surface at a depth $= \frac{2}{3}$AB below A, will be likewise known.

Similarly, if through the points c_3'', c_4'', c_5'', &c., straight lines be drawn making an angle $= (90 - \theta)$ with BD″, that is, parallel to BG″, and these straight lines be produced to meet the corresponding planes of cleavage produced in the points 3″, 4″, 5″, &c., then will the sides of the series of triangles so formed represent on the scale of loads already determined, the several forces keeping the corresponding wedges of earth at rest at the moment when motion is *just about* to take place *up* the corresponding plane of cleavage; and the triangle OKR″, *Fig.* 101, *Plate* XXV., may be regarded as representative of any one of them.

301. Now it may be proved that the points 1′, 2′, 3′, &c., lie on an hyperbola, which touches AB at B, and cuts the surface AD at D. A geometrical proof will be found in Chalmer's "*Graphical Determination of Forces in Engineering Structures*," p. 303, but, as the result is not of any practical importance, no proof is offered here. Similarly, the points 1″, 2″, 3″, &c., may be shown to lie on an hyperbola the asymptote of which is BG″ produced.

If these curves be drawn in, it will be seen that the position of the

plane of cleavage of maximum passive pressure (P') and the magnitude of that pressure are determined by finding the point in which a tangent to the lower curve, drawn parallel to BD, meets it; while the plane of minimum active pressure (P'') and the magnitude of that pressure are determined in a similar manner by finding the point in the upper curve at which the tangent, drawn parallel to BD″, meets it.

These planes and pressures are indicated in *Fig.* 104, *Plate* XXVII., by thick link lines ; both planes of cleavage lie near plane BC_4.

302. But the positions of these planes of cleavage may be graphically determined at once as follows :—

The expression $x = c \pm \sqrt{c(c-a)}$ of para. 277 admits of graphic representation thus. Through F (or F″), *Fig.* 102, *Plate* XXVI., draw Fl (or F″l) parallel to DD″. (The straight line FF″ is necessarily parallel to DD″ owing to symmetry, para. 276). Then Bl represents $(c - a)$, and BG (or BG″) represents the expression c. If, therefore, a semicircle be described on BG (or BG″) and at l the perpendicular lm be erected, then will $Bm = \sqrt{c(c-a)}$. If a circle be described with centre B and radius Bm to cut BG (or BG″) produced if necessary in n and k, and straight lines be drawn through these points parallel to DD″, then will the points E and e'' be determined, and consequently also C and c,″ and the required planes of cleavage BC and Bc'' be known.

303. It is generally, however, more convenient to make the necessary construction on the surface line DD″, instead of on the straight lines BG or BG″ (compare para. 276).

Since $x : c : a$ :: CE (or $c''e''$) : BG (or BG″) : AF (or AF″),
:: DC (or D″c'') : DG (or D″G″) : DA (or D″A),
∴ $x : c : c - a$ :: DC (or D″c'') : DG (or D″G″) : AG (or AG″),
and the expression for finding the plane of cleavage of maximum passive resistance, viz. :—

$$x = c - \sqrt{c(c-a)}$$

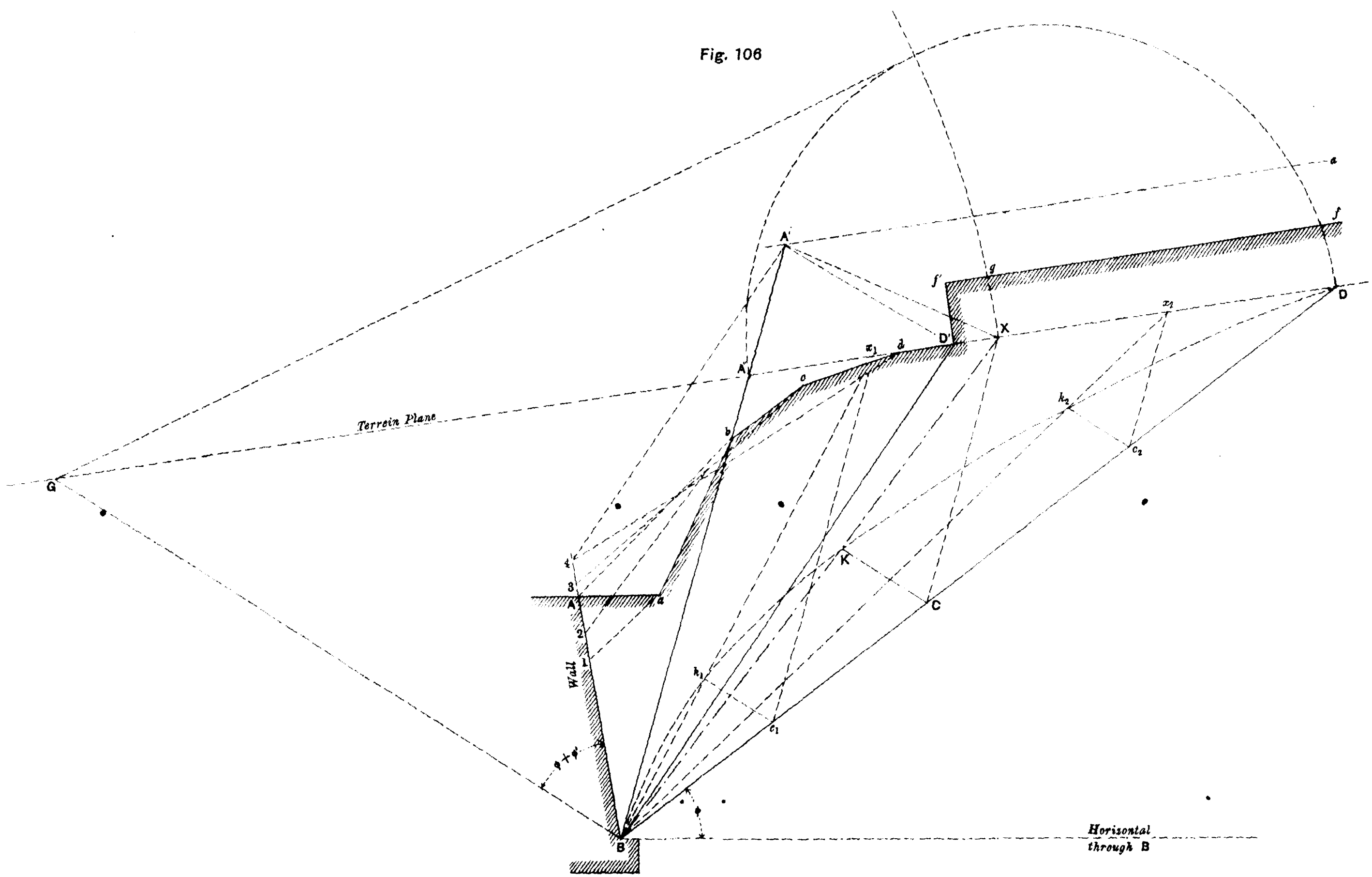
becomes, when transformed,

$$CD = DG - \sqrt{DG \times GA}$$

the geometrical construction of which is evident and shown in *Figs.* 102, 104 and 106. Similarly, the expression for finding the plane of cleavage of minimum active pressure, viz.,

$$x = c - \sqrt{c(c-a)}$$

Fig. 106



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becomes, when transformed,

$$D''c'' = D''G'' + \sqrt{D''G'' \times G''A}$$

the geometrical construction of which is likewise evident, and shown in *Figs.* 102 and 104.

304. The planes of cleavage having been determined by the above simple construction, the value of the corresponding conjugate pressure is at once known by the method indicated in para. 300 and shown in *Fig.* 104.

Thus, suppose the plane BC of maximum passive resistance P′ has been determined. Through C, *Fig.* 102, draw the vertical Cc' to meet BD in c', and from c' draw $c'y$ parallel to BG to meet BC in y. Then will the length of $c'y$ represent the maximum passive resistance P′ on the same scale as Bc' represents the weight of the wedge of earth ABC. This may be calculated and the scale, therefore, determined.

In a similar manner the magnitude of the minimum active resistance P″ corresponding to the plane of cleavage Bc'' may be determined.

305. If it be required to draw a curve the abscissæ of which, on any given scale of loads, represent the total resultant pressures acting over the corresponding depths of the vertical area AB, which are represented on any given lineal scale by the ordinates, we know from para. 277 that $P \propto (\text{depth})^2$, and acts at $\frac{2}{3}$ depth below the surface. If, then, for P we write x, and for the corresponding depth we write y, the law of the curve is represented by the expression $x \propto y^2$, which is the law of variation of a parabola whose apex is at A and axis coincident with the surface. One or two values of P′ or P″, as the case may be, being either calculated or determined graphically by methods already described, the curve may be drawn in the manner explained in para. 181 of Vol. I. But it must be remembered that the resultant pressure, to which the value of x corresponds, really acts at a depth $= \frac{2}{3}\,y$.

CHAPTER XIX.

STABILITY OF RETAINING WALLS TO RESIST THE PRESSURE OF EARTH.

306. In inquiring into the stability of a retaining wall, it is necessary to see that the conditions of stability required in an ordinary wall are fulfilled. These have been already investigated, and are as follows :—

1*st.* *The line of resistance must intersect every bed joint, sufficiently far within the outer edges to prevent any risk of their crushing.*

2*nd.* *The angle which the resultant pressure at any bed joint makes with a normal to that joint must not be greater than the angle of which* $\frac{4}{5}$*ths of the co efficient of friction of the material is the tangent.*

It is also advisable in every case to inquire both into the stability of the wall itself as regards the most dangerous bed joint in the masonry and into the stability of the wall as regards the earth on which it stands.

In designing a wall for any particular case, the simplest method is—

1*st.* To determine what form of section the wall is to have.

2*nd.* Assuming certain dimensions, to inquire whether they fulfil the conditions of stability.

307. Before, however, examining the different forms which sections of retaining walls may take, it is necessary to consider the problem of how the stability of the retaining wall should be measured, under the most general conditions that are likely to occur.

For hitherto our investigations have been confined to the examination of the state of stress obtaining in a loose mass of earth, bounded above by a sloping plane surface ; and we have seen that in such a mass the pressure on any plane parallel to that surface is vertical and of a uniform intensity equal to the weight of the vertical prism which stands on unity of area of the given plane ; also that the stress, if any, on any vertical plane dividing the mass is parallel to the surface. But

these conditions are necessarily not applicable to masses of earth whose surfaces are irregular, including, for instance, cases of surcharges, ramparts, and buildings, situated near the retaining wall, the pressure on which cannot necessarily be parallel to the surface of the mass. For these reasons a graphical solution of the general problem is given, taken from Chalmer's "*Graphical Determination of Forces in Engineering Structures*," p. 301.

The general method of treatment is quite similar to that already described for the simpler case of uniform surface slope.

308. The reader must, however, at once clearly realize that *what we have hitherto called the maximum resultant passive conjugate resistance* P′ *to pressure from without, when treating of internal stress in the mass of earth, measures also the magnitude of the maximum active pressure which the mass of earth exerts* (in an opposite direction to P′) *against the wall built to retain it.*

Similarly, *what we have hitherto called the maximum resultant active conjugate thrust* P″ *against resistance from within the mass, measures also the maximum passive internal resistance with which the mass of earth can resist pressure from without* (exerted, of course, in an opposite direction to P″).

Now the limiting directions of these resultant pressures acting against the wall's interior surface (*i.e.*, the surface in contact with the earth) are evidently determined by the value of the angle of repose of the earth on the material of the wall, which call ϕ', for it is at this angle that the resultant earth pressure must incline to the normal to the wall's interior surface either above or below it at the moment when the earth is just about to slide on the wall, or the wall on the earth. Either of these directions is physically possible owing to slight displacement of the earth or wall.

Since vertical subsidence or crushing of the material of the wall is the principal danger to be guarded against, it would seem that the most effective direction of the resultant earth pressure, when considered as an *active* force, applied to the structure, is that which tends in the direction of gravity; while, on the other hand, when considered as a *passive* force, resisting such tendency to subsidence, its most effective direction is that which opposes gravity.

Hence the active equivalent of P′ should act above, and the passive

equivalent of P″ below, the normal drawn to the surface of the structure which is in contact with the earth.

309. Now, as in the more simple case of uniform surface slope, so in this more general case, it will be found convenient to determine the value of P′ or P″ as the case may be, by supposing the representative triangle of forces OKR′, or OKR″, of *Fig.* 101, to be revolved to the right, in the direction of the hands of a watch, through an angle of $90 - \phi^\circ$ in the case of P′, and of $90^\circ + \phi^\circ$ in that of P″, so that the planes of repose BD, or BD″, may represent the direction of the weight of the earth wedge in the two cases, and it will be, further, found convenient, as before, to lay off the direction of P′ or P″, as the case may be, relatively to that of BD instead of BA. A little consideration will show that this direction makes an angle of $(\phi + \phi')$ in the case of P′, measured to *left* of AB, and of $(\phi + \phi')$ in the case of P″, measured to the *right* of AB. This is at once evident from *Fig.* 105.

For, suppose angles measured to the right of AB to be reckoned positive and those to the left negative. Before revolution in the direction of the arrow, the active equivalent of P′ produces its greatest effect when inclined at $(90 + \phi'^\circ)$ to the left or negative side of AB. If then this direction $-(90 + \phi'^\circ)$ be revolved in the positive direction through an angle $(90 - \phi^\circ)$, the resulting inclination to AB will be $-(90 + \phi') + (90 - \phi) = -(\phi + \phi')$, or that angle measured to the *left* of AB.

Again, the most effective direction of the passive equivalent of P″ is inclined at $(90 - \phi')$ to the left, or negative, side of AB. If this negative angle be revolved in a positive direction through the angle $(90 + \phi)$, the resulting direction will be inclined at $-(90^\circ - \phi') + (90^\circ + \phi) = +(\phi + \phi')$ to AB, or that angle measured to the *right* of AB.

The corresponding inclinations of the least effective forces are evidently formed by adding $2\phi'$ in the case of P′, and deducting the same angle in the case of P″, giving $(-\phi + \phi')$ in the first case, and $(\phi - \phi')$ in the second.

*Graphical Solution of the General Problem regarding the Stability of Retaining Walls to resist Earth Pressure.**

310. Let AB, *Fig.* 106, *Plate* XXVIII., represent the back of a retaining wall built to resist the pressure of a mass of earth, whose surface,

* Chalmer's Graphical Determination of Forces, p 301.

extending from A to the point D, where the plane of repose BD meets the upper surface, is of the form A$abcd$D′, and suppose that the upper surface from the point D′ and onwards is loaded with a uniform vertical load (that of a wall say). Let DD′$f'f$ represent the parallelogram of earth of unit thickness, whose weight equals that of the wall. Then the figure BA$abcd$ D′$f'f$ DB may be regarded as a graphic representation both of the weight and figure of the mass of earth lying above the plane of repose BD.

It is first necessary to reduce the irregular figure BA$abcd$D′B to a triangle of equal area. This is done by the usual method, the steps of which are, for convenience, indicated.

Join A and b; draw a1 parallel to Ab; A1b = Aab; hence remainder Aa1 = remainder a1b; hence figure B1bB = figure BAabB.

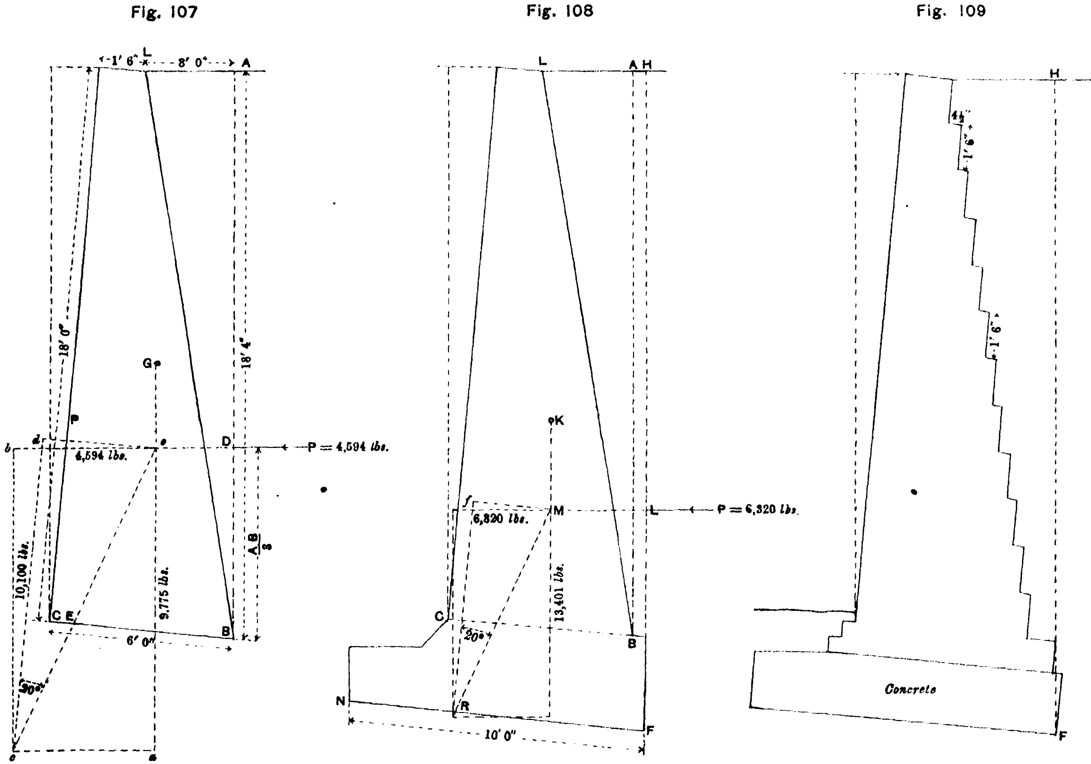
Join 1 and c; draw b2 parallel to 1c; then B2cB = B1bcB = BAabcB.

Join 2 and d; draw c3 parallel to 2d; then B3dB = B2cdB = B1bcdB = BA$abcd$B.

Join 3 and D′; draw d4 parallel to 3D′; then B4D′B = B3dD′B = B2cdD′B = B1bcdD′B = BA$abcd$D′B.

In order to adapt the representative triangle so as to include the parallelogram D′$f'f$D, draw the straight line aA′ parallel to D′D, and at a distance = 2D′f' from it. Through 4 draw 4A′ parallel to BD′ to meet A′a in A′. Then the triangle BA′D′B = triangle B4D′B = figure BA$abcd$D′B.

The plane of maximum passive resistance can now be determined in the manner already explained thus:—Draw BG making an angle $(\phi + \phi')$ with AB, ϕ being equal to the angle of repose of the earth on itself, and ϕ' that of the earth on the masonry of the retaining wall along the surface AB, and produce BG until it meets the plane of the surface (Dd) in G. If A″ be the point in which the surface plane Dd cuts BA′, then a mean proportional GX to the lengths GA″ and GD will determine the point X, in which the plane of rupture BX meets the terrain plane. If through X, the straight line XC be drawn parallel to BA′, to meet BD in C, and through C the straight line CK be drawn parallel to BG, to meet the plane of rupture in K, then will CK represent the maximum passive resistance on the same scale that BC represents the weight of the earth prism BA$abcd$D′$f'g$XB, that is, of the equivalent earth prism BA′XB which may be calculated and the



Scale for Wall = 4 Feet to 1 Inch.
Scale for Forces in Fig. 107 = 4,000 lbs. to 1 Inch.
Scale for Forces in Fig. 108 = 8,000 lbs. to 1 Inch.

Litho. T. C. Press, Roorkee.

CHAPTER XX.

REMARKS ON RETAINING WALLS.

312. It will be observed that the graphical method described in the preceding Chapter measures the stability of the retaining wall *in the limiting position*, that is, at the moment when the wall is just on the point of slipping away from the earth behind it, or the earth from the wall, owing to settlement at the foundations or from some other cause. Simple cases of walls, however, required to retain masses of earth the slope of whose surface is uniform, cases of "relieving arches" (Example II. following) and of underground arches, such as occur in vaults and tunnels, are more conveniently dealt with by applying the results of Chapter XVI., as illustrated in the two examples following. In thus applying these results to measure the stability of a retaining wall at any joint, the weight of earth resting on the inner face of the wall above that joint is considered as adding to the stability of the wall;* and in the case of "relieving arches" and of underground arches, the earth lying on the extrados is so considered. For instance, in Example I. following, *Figs.* 107 and 108, the weight of the prism of earth LBA is considered as adding to the stability of the wall above the joint CB.

When, however, the back or inner face of the wall rests on the earth, as in *Figs.* 114, 116 and 117, the part of the masonry cut off by the vertical plane passing through the heel of the wall is considered as adding to the stability of the wall only to the extent of the difference between its weight and that of a prism of earth of equal volume,* so that if the earth be of the same specific gravity as the masonry of which the wall is built, the material of the part of the wall lying behind the said vertical would add nothing by its weight to the stability of the wall, while, were it of less specific gravity, its presence would be positively injurious. It will thus be seen that in a leaning wall the materials are not employed to the best advantage.

* Rankine's Applied Mechanics, page 250, and his Civil Engineering, page 402.

Example I. following is selected by permission from Cols. Wray and Seddon's "Instruction in Construction," 3rd Edition, page 873, and affords a good and complete illustration of the method above described.

313. *Example I.*—It is required to design a retaining wall of brickwork in mortar, to sustain a bank of earth, whose surface is level with the top of the wall and horizontal, the bricks and mortar having an average resistance to crushing, &c., as follows:—

Weight of brickwork to be taken at 100 lbs. per cubic foot.

Ultimate resistance of brickwork in mortar to crushing, 600 lbs. per square inch, or 39 tons per square foot; *factor of safety,* 8.

Ultimate resistance to cracking, 20 tons per square foot, or, say, 320 lbs. per square inch; *factor of safety,* 3.

Co-efficient of friction of brickwork, the mortar being moderately fresh, 0·74.

Weight of earth, 110 lbs. per cubic foot.

Angle of repose of earth behind wall, 37°; of the earth on which the wall stands, 25°.

Pressure on earth below the wall not to exceed $1\frac{1}{4}$ tons per square foot, the soil being sandy gravel.

Factor of safety against sliding not to be less than 1·2, both as regards the masonry joints, and the wall on its base.

Height of wall above footings to be 18 feet.

There being no reason why the wall at top should not be built as thin as good construction will allow (*i.e.*, having sufficient width for a solid coping), the *wall's top will be taken at two bricks thick.*

Face batter to be taken at $\frac{1}{12}$, or 18 inches.

Wall above Foundations (*Fig.* 107, *Plate* XXIX.).—Assume for trial a thickness at the top of the footings of one-third the height, *i.e.*, 6 feet, which will be eight bricks thick, and will allow of the back of the wall being built with offsets $4\frac{1}{2}$ inches at every 18 inches in height, *i.e.*, at every six courses of brickwork.

Since the moment of the conjugate pressure on the vertical plane AB, *Figs.* 107 and 108, increases as the cube of the height, while the moment of resistance increases very little faster than the height, the wall must be more likely to overturn at the joint at the top of the footings than at any other. At this joint, therefore, the centre of pressure will, in approaching the edge of rotation, gradually concentrate the pressure at that edge;

so that the same joint which is dangerous as regards overturning, will also be dangerous as regards crushing.

The mere question of overturning, which only requires the line of resistance to cut every bed joint, need not be gone into, it being covered by that of safe resistance to crushing, to insure which the centre of pressure must fall sufficiently within the edge of maximum compression to keep the latter within safe limits. It is therefore sufficient to enquire into the stability of the structure as regards crushing, and at the joint BC where it tends to overturn.

Moreover, as the pressure tending to produce sliding increases as the square of the height, while the weight resisting it increases very little faster than the height, this same joint BC must be that on which the tendency to slide is greatest.

Then considering this bed joint BC for a unit of the wall 1 foot in length—

Weight of wall above the bed joint BC	= 6,750 lbs.
" earth resting on wall, *i.e.*, in front of the plane AB, ...	= 3,025 lbs.
Total weight on BC, ...	= 9,775 lbs. = W.

The centre of gravity of the mass is found to be at G, *Fig.* 107.

Then, the earth surface being horizontal, the pressure on the vertical plane AB $= \frac{wh^2}{2} \cdot \frac{1 - \sin\phi}{1 + \sin\phi} = 4{,}594$ lbs. which acts at D, BD being $\frac{1}{3}$ AB.

Through G draw a vertical, and through D a horizontal, intersecting in O. On any scale of parts, make $Oa = W = 9{,}775$ lbs., and $Ob = P = 4{,}594$ lbs. Complete the parallelogram; then Oc represents the resultant pressure on BC, and E is the centre of pressure. CE will be found to measure $9\frac{1}{2}$ inches.

Resolve Oc normal and parallel to BC, then $cd = 10{,}100$ lbs. is the normal pressure on BC; which is spread over a surface equal to three times CE multiplied by the length of the section of the wall under consideration, which is one foot, and the maximum pressure at C is equal to twice the mean pressure.

Hence the pressure per square inch at C $= \frac{2 \times 10{,}100}{3 \times 9\frac{1}{2}'' \times 12''} = 59$ lbs.

This, giving a factor of safety of over 10 as regards crushing, and over 5 as regards cracking, somewhat more than fulfils the required condition of stability; if it had not fulfilled the condition it would have been necessary to assume a greater thickness for BC, and recalculate.

Next, measure the angle *Ocd;* which is found to be 20°, the tangent of which is ·36; so that, as the co-efficient of friction of this joint is given at ·7, the factor of safety is greater than required, being nearly 2, instead of 1·2.

The wall itself is therefore stable as regards overturning, crushing, and sliding.

Footings and Foundations.—Assume the general outline of the footings and foundations to be as shown in *Fig.* 108, the underside of the foundation being, at its outer end, about 3 feet underground, so as to get below the effects of frost.

Draw a vertical FH through the back of the foundation. The weight of the wall and earth in front of this plane is found to be 13,401 lbs., and the centre of gravity of the mass is at K.

The pressure on the plane FH = 6,320 lbs., acting at L, at a distance from F = $\frac{1}{3}$ FH.

Compounding the pressure and the weight, as before, MR represents the resultant pressure on the base of wall FN, and the wall is safe against overturning, since the centre of pressure, R, falls within the base.

The normal pressure on NF, which is 10 feet wide, is represented by *f*R = 14,000 lbs.; therefore, if spread uniformly over NF, the pressure would = 14,000 lbs. ÷ 10 feet = 1,400 lbs. per foot superficial; but as NR = $\frac{1}{3}$ NF, nearly, the intensity of the pressure at F may be taken as *nil*, while that at N = twice the mean pressure = 2,800 lbs., or 1¼ tons, which is the safe limit laid down. If the maximum intensity of pressure exceeded 1¼ tons it might compress the soil at N, causing the wall to lean forward from its original position, which would have to be guarded against by lengthening the toe of the foundation sufficiently to reduce the intensity of pressure within the safe limit, care being taken to make the projecting toe thick enough to prevent its breaking off under the pressure.

Finally, the angle MR*f* is 20°, the tangent of which is ·36; and the co-efficient of friction of the soil on which the wall stands is ·47; so

that there is a factor of safety, as regards sliding forward, of 1·3, which is greater than 1·2.

If there were any doubt as to the condition of the earth below the wall, as regards moisture, it would be better to incline the concrete backwards.

The wall may now be designed as shown in *Fig.* 109, *Plate* XXIX.

Example II.—"*Relieving Arches*", *or* "*Revêtements en Décharge*".—When the front vertical wall is "relieved" of the earth pressure by a series of arches, as shown in *Fig.* 121, this arrangement is known as "relieving arches" or "revêtements en décharge"; the arches may be arranged in one tier, or in many tiers, and their front ends may be closed by a vertical wall, which thus presents the appearance of a retaining wall, although, as the length of the archway is such as to prevent the earth from abutting against the wall, the latter is thus relieved of pressure.

Professor Rankine in his *Civil Engineering*, p. 413, gives the following formulæ for computing the length of a relieving arch from its clear height, or its clear height from its length :—

If d be the depth of the crown of an arch below the surface, h its height, l its length, and ϕ the angle of repose of the earth, then

$$l = \cot\phi \left\{ h + \frac{d}{(1+\sin\phi)^2} \right\}$$

$$h = l \tan\phi - \frac{d}{(1+\sin\phi)^2}$$

The earth may, as a rule, be filled in solid up to the back of the front wall, which, if properly joined to the piers and arches, should be strong enough to resist the pressure of the earth enclosed behind it; the more the arches are loaded, the greater will be the stability of the entire structure.*

The resultant pressure on the base of the structure may be found in the usual way by combining the pressure on the vertical plane OD, *Fig.* 121, with the weight of the combined mass of masonry and earth OAED, lying in front of that plane.

In soft ground the bases of the piers of the lowest tier of relieving arches should be connected by means of inverted arches, so as to distribute the pressure over the whole area covered by the structure.

Example III.—*Buttressed Horizontal Arches.*—*Figs.* 110 and 122

* Cols. Wray and Seddon's "Instruction in Construction," 3rd Edition, page 270.

show the ground plan or horizontal section of portions of two rows of buttresses, connected by horizontally arched walls, the tops of the buttresses shown in *Fig.* 122 being further connected by arches to support a platform or surcharged bank of earth.

Referring to *Fig.* 110, Professor Rankine gives the following rule for finding the length, T = DE, of the buttress in terms of the thickness which would be required for a wall of rectangular section strong enough to sustain the same bank of earth; to be calculated as in Example I. Thus, if l = AB = the distance from crown to crown of the horizontal arches connecting the buttresses = the breadth of the mass of earth which would press on the hypothetical rectangular wall shown by dotted lines in *Fig.* 110.

t = the thickness of the hypothetical wall.

h = the height of the retaining wall, and of the buttresses.

b = the thickness of each buttress.

w = the weight per cubic foot of both wall and buttress.

Then, we have for the weight of the buttress, $B = whTb$, and for that of the hypothetical wall $W = whtl$; so that, taking moments about D we have

$$whTb \times \frac{T}{2} = whtl \times \frac{t}{2}$$

whence $$T = t\sqrt{\frac{l}{b}}, \text{ and } b = l\frac{t^2}{T^2}$$

In soft ground the bases of the buttresses might be connected by means of inverted arches, to distribute the pressure.

*General Remarks.**

314. *Section of the Wall.*—*Figs.* 111 to 122, show sections of walls which have actually been built. Those shown in *Figs.* 118 to 120 are counterforted; *Fig.* 121 shows a wall with "relieving arches"; *Fig.* 122 shows a wall consisting of buttressed horizontal arches, arched at top.

When a wall is only called on to sustain the pressure of a bank of earth, or of that of the earth together with the building on it, its thickness at top need not be more than is sufficient for an efficient coping.

* These remarks, together with the Section of Walls shown on *Plate* XXX., are taken by permission from Cols. Wray and Seddon's "Instruction in Construction," 3rd Edition, pp. 369, 371 *et seq.*

When, however, the wall is liable to shocks, as when a road runs behind it, or as in the case of reservoir dams, or of lock walls, the thickness at top should be considerable.

It is obvious that the wider the base of the wall for a given weight of material, the greater will be the quantity of earth adding to its stability, so that it is always advisable to reduce the thickness at top to a minimum, consistent with sound construction; experience, moreover, shows that a width of base equal to one-third the height of wall will nearly in all cases ensure stability.

Batters, both straight and curved.—It is generally agreed among Engineers that the batter base should not exceed $\frac{1}{6}$th of the height of the wall; in practice a batter of from $\frac{1}{2}$ inch to 2 inches per foot is generally adopted, a greater inclination being found to afford too great a facility for the entrance of water into the joints; and as the lowest part of the wall is that most liable to injury from this cause, it is evident that the face of a curved batter should be a tangent at the foot to a plane having an inclination not greater than $\tan^{-1} 6$, the height being six times the base, as in the section shown in *Fig.* 123.

As regards the advantage in point of stability of either a straight or curved batter in comparison with a vertical face, it may be said that the battered wall is more stable against sliding forward, either on any of the bed joints or on its base, that is, if the joints are at right angles to the face of the wall, which is the usual arrangement. A batter has the advantage of *apparent* stability; for if the wall be originally vertical, and rotate ever so little, it offends the eye, and gives the idea of instability; whereas, if the wall has a batter originally, the alteration in inclination is not apparent.

Curved batter walls are more difficult and expensive to construct, and unless a good deal of material be cut to waste, the joints must be thicker at the back than at the front of the wall, leading to unequal settlement. They are, however, a good deal used.

All walls should have some slight batter, when circumstances admit of it, the straight being preferred to the curved batter, and in the case where the fining down of the material is a matter of great importance, alternative designs should be prepared.

Wing walls of Bridges, carrying railways over roads, which have to sustain the pressure of the railway embankment, are frequently made

curved on plan, which is an advantageous form for resisting the pressure of the embankment.

Counterforts.—Professor Rankine shows that in vertical rectangular walls, a small saving of masonry is effected by the use of counterforts, but their efficiency depends upon the care with which they are bonded into the wall. The form of counterfort which appears likely to be strongest is that shown in *Fig.* 120, because the resistance to fracture at the counterfort either along AB or BC increases with the pressure from top to bottom of the wall, the base of the counterfort being made as large as possible with a given quantity of material, and therefore having (as compared with a wall without counterforts) an increased amount of earth tending towards the stability of the wall.

*Constructive Details.**—The angle of repose of the earth, together with its weight, should be ascertained by experiment, the earth being in that state in which, by proper drainage, it can be ensured to remain.

The maintenance of a thoroughly good bond is of the greatest importance in building retaining walls; their backs should be left rough, and if built in steps, the steps should be as numerous as possible. The number of steps or offsets will be governed by the slope of the back of the wall, each offset, in brickwork, as a rule projecting half a brick's length.

A layer, at least 12 inches thick, of loose stone, gravel or other porous material, to facilitate the passage of the water to the loop-holes, should, especially in retentive soils, be packed up behind the walls. In most cases the loop-holes should be numerous, to prevent any accumulation of water in the earth at the back of the wall. With loose packing at the back of the wall, as described, the weep-holes should be placed along its foot at from 5 feet to 10 feet apart, according to the nature of the case.

Professor Rankine in his *Civil Engineering*, para. 140, gives the proportion of one weep hole to every 4 square yards of wall face, but they would not be distributed over the surface of the wall, except when a retentive soil or some impervious layer butts against it.

Weep-holes should not be under 7 square inches in sectional area, or they will soon choke, and will not be easily cleared; they are often pointed and lined with cement; if pipes are used, they should be pointed in cement.

* These details are taken from Cols. Wray and Seddon's "Instruction in Construction," p 372.

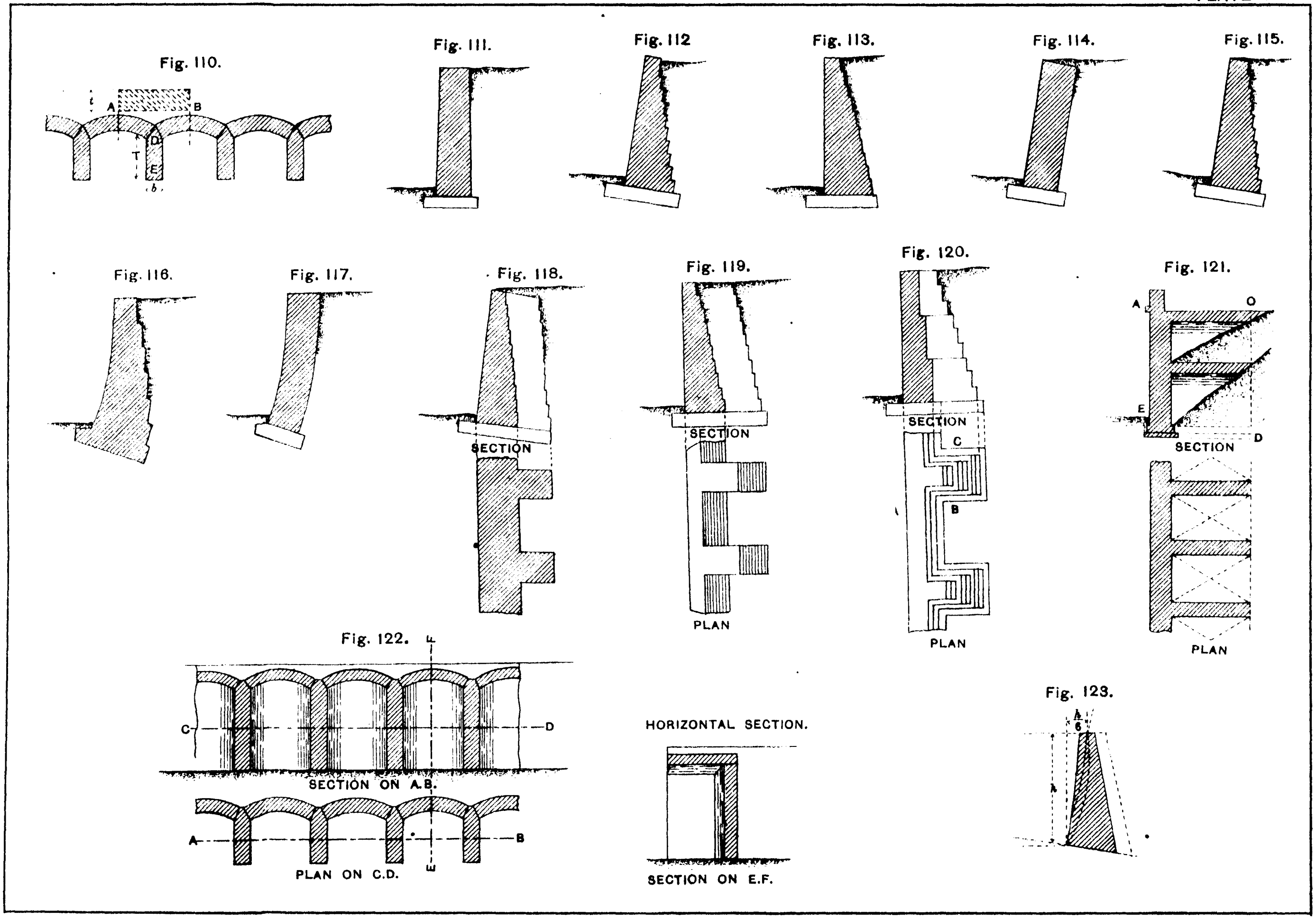


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TABLE B.

TRANSVERSE STRENGTH OF CONCRETE AND OTHER BEAMS, SUPPORTED AT ENDS.

No.	Composition. Portland Cement.	Sand.	Aggregate.	Age in days	Breadth. In.	Depth. In.	Clear Span. In.	No. of Tests.	Loaded at	Average Breaking Weight. Cwt.	Reduced to Breaking Weight at centre. Cwt.	One-half Weight of Beam between supports. Cwt.	Total Central Load. Cwt.	Constant. s	Average Constant. s	Authority.
1	1	..	1 Coke breeze, ...	7	3	5	72	1	Centre, ..	3·85	3·85	0·31	4·16	5·99	..	A
2	1	..	2 Crushed brick, ...	6	12	8	60	1	„ ..	13·23	13·23	1·67	14·9	1·74	..	A
3	1	2c		90	„	12	36	2	Central 6". ..	155	142·06	1·55	143·61	4·48	3·88	B
4	1	2d		„	„	„	„	3	„ ..	113·33	103·88	1·55	105·43	3·29		
5	1	..	4 Clean breeze, ...	43	30	6·5	59	1	Central 16", ..	66·32	57·32	2·12	59·44	4·15	3·63	C
6	1	..	4 Broken brick, ...	„	„	6	59·5	1	„ ..	40·52	35·07	2·53	37·6	3·11		
7	1	0	5 Shingle, ...	139	12	12	36	1	Centre, ..	85·62	85·62	1·77	87·39	2·73	..	D
8	1	1	5 „	„	„	„	„	1	„ ..	68·74	68·74	1·91	70·65	2·21	..	D
9	1	2	5 „	„	„	„	„	1	„ ..	43·61	43·61	1·84	45·45	1·42	..	D
10	1	3	5 „	„	„	„	„	1	„ ..	27	27	1·76	28·76	0·89	..	D
11	1	2	6 Gravel, ...	90	12	12	36	3	Central 6", ..	46·67	42·78	„	44·54	1·39	1·39	B
12	1	2	2 Broken stone 1½", 4 „ „ 3½", ..	„	„	„	„	3	„ ..	52·5	48·12	„	49·88	1·56		
13	1	2	6 „ „ 1¼", ...	„	„	„	„	3	„ ..	40·83	37·43	„	39·19	1·22		
14	1	0	9 Shingle, ...	95	12	12	18	1	Centre, ..	83·06	83·06	·77	83·83	1·31	1·27	D
							36	1	„ ..	38·33	38·33	1·53	39·86	1·24		
15	1	1	8 „	„	„	„	18	1	„ ..	71·18	71·18	0·92	72·10	1·12	1·18	D
							36	1	„ ..	38·33	38·33	1·84	40·17	1·25		
16	1	2	7 „	„	„	„	18	1	„ ..	56·16	56·16	0·94	57·10	0·89	0·94	D
							36	1	„ ..	30·15	30·15	1·88	32·03	1·00		
17	1	3	6 „	„	„	„	18	1	„ ..	30·28	30·28	0·93	21·21	0·49	0·58	D
							36	1	„ ..	20·04	20·04	1·86	21·90	0·68		
18	1	3	6 „ screened, ...	28	21	9	99	1	Central 6", ..	5	4·85	6·57	11·42	1·00	0·92	E
							45	1	Central 12", ..	20·88	18·35	3·08	21·43	0·85		
							45e	1	„ ..	13·6e	11·92	„	15·	0·59e		
19	1	4	5 „	95	12	12	18	1	Centre, ..	24·91	24·91	·84	25·75	0·40	0·59	D
20	1	4	5 „	139	„	„	36	1	„ ..	23·71	23·71	1·69	25·4	0·79		
21	1†	1	½ Gravel ¾", ... 3 Broken stone (1" to 4"), ..	182	6	12	36	1	Central 12", ..	72	66·05	·75	66·8	4·17	..	F
22	1‡	1	.. ..	100	2	2	4	2	Centre, ..	5·22	5·22	a	5·22	3·91	..	G

NOTES TO TABLE.

The weight of the beam itself is part of the load, and must, therefore, be considered in the calculations, otherwise grossly inaccurate results would sometimes be obtained; *e.g.*, the weight of the first beam of the three numbered 18 is nearly three times as much as the load put upon the beam. The weight of the beam must really be considered as a distributed load, and as the stress of a distributed load is only one-half that of a central load, one-half the weight of the beam is given in the column.

† Stag Cement. ‡ Roman Cement.

a. Weight too small to be considered. *b.* Presumably seven days; the various ages of the beam should be carefully noted. *c.* Coarse sand. *d.* Fine sand. *e.* Part of a larger beam which fell before it had properly set, and was, therefore, probably strained.

Authorities.—A David Kirkaldy. B John Kyle. C Col. Crozier. D Darnton Hutton. E C. Colson. F Wm. Kidd. G Q. A. Gilmore.

Along the foot of the retaining wall there should be a surface drain to carry off the water from the weep-holes, which otherwise might sink into the ground beneath the wall and interfere with its stability.

Made earth behind the wall should be deposited in thin layers, inclined slightly towards the wall for drainage purposes, and rammed not too heavily; some Engineers on the other hand prefer to ram the layers well, but to stop the ramming within a short distance of the wall. The earth backing should in no case be tipped in. If the backing be of clay, care should be taken that it is not too dry, otherwise it will swell on becoming moist and produce increased pressure against the wall.

If the earth retained be of such a nature that with excess of water it become mud, and if it be impossible by drainage to prevent excess of water, it is best to design the wall to retain a fluid having the weight of the mud, and in such cases an apron of stout sheet piling along the toe of the wall will prevent the squeezing out of the subsoil; or a bank of rubble or gravel may be placed against the back of the wall in such a way that the angle of repose will not be affected by the mud; the pressure will then be reduced to that of the bank, or at worst, to that of water up to the drainage level.

Hoop-iron is worse than useless in retaining or any other kind of walls, unless it be carefully protected from oxidation and laid in strong hydraulic mortar or cement. Moisture causes the iron to rust and expand with a force sufficient to open the joints of the work. When properly protected, however, hoop-iron may be most valuable as a bond.

Professor Rankine in his *Civil Engineering*, p. 381, lays down the rule that the pressure on foundations in firm soil should not exceed from 2,500 to 3,500 lbs. per square foot; and it is a common rule to limit it to one ton per square foot, which, however, is a low limit, unless the foundations be weak or unsound.

CHAPTER XXI.*

SOME REMARKS ON FOUNDATIONS.

315. The question of Foundations, both on land and under water, is fully considered in Vol. I. of the "Roorkee Treatise on Civil Engineering", the following remarks, having reference principally to concrete in foundations, should be read in conjunction with what is stated in the Volume referred to.

316. A layer of concrete is now the usual *bed of the foundations* in homogeneous and slightly compressible soils. The breadth is usually about 1 foot greater than the thickness of the brickwork footing above it, *i.e.*, an offset of 6 inches is left at either side, so that after the concrete bed has been laid the exact positions of the brickwork footings may be accurately marked on it, and a good margin for bearing left beyond. The London County Council regulations give 4 inches as the projection beyond the footings of the concrete, which is also specified to be 9 inches thick, irrespective of the height above. Such regulations, however, are open to the objection that no reference is made to the superincumbent weight which should determine both the breadth and thickness of the concrete.

317. It may be that owing to concrete being cheaper than brickwork, footings of the latter may be entirely dispensed with and the concrete taken up to ground level. Supposing, however, that brickwork footings are used, the question arises, what should be the *breadth* and *thickness* of the concrete bed ?

As regards the breadth there are two considerations, (1) the practical width to allow a fair base for the superstructure, and (2) such a width as will spread the weight over such an area as will bring the weight per square foot on the soil below within safe limits.

As regards (1) the thickness of the walls of a dwelling house are determined by questions of comfort rather than of stability.

* These remarks are taken by permission from a paper by Major Scott-Moncrieff, R.E., Instructor in Construction at the School of Military Engineering, Chatham

The rules adopted in England (London Building Act) are given in "Hurst's Pocket-book," and specify that whatever the thickness of the walls the footings should have a base equal to double that thickness. An offset of even 4 inches to the concrete bed beyond the footings will usually give more than is necessary to satisfy the requirements of the weight on the soil being within safe limits. Those limits were formerly stated by Professor Rankine to be 1 ton to $1\frac{1}{2}$ tons per square foot on ordinary soils, but experience has since shown that much greater weight can safely be brought to bear on most soils.

318. Mr. Newman, author of several books on Foundations and kindred subjects, gives the following Table :—

TABLE A.

Description of Earth.	Approximate safe maximum load in tons per sq foot.
Bog, morass, quicksand, peat moss, marsh land, silt, ..	0 to 0·20
Slake and mud, hard peat turf, ...	0 to 0·25
Soft, wet, pasty or muddy clay and marsh clay, ...	0·25 to 0·33
Alluvial deposits of moderate depths in river beds, &c, ..	0·20 to 0·35
NOTE.—When the river bed is rocky and the deposit firm they may safely support 0·75 ton, but not more.	
Diluvial clay beds of rivers, ...	0·35 to 1·00
Alluvial earth, loams and loamy soils (clay and 40 to 70 per cent. of sand) and clay loams (clay and about 30 per cent. of sand), ...	0·75 to 1·50
Damp clay, ...	1·50 to 2·00
Loose sand in shifting river bed, the safe load increasing with depth, ...	2·50 to 3·00
Upheaved and intermixed beds of different sound clays, ..	3·00
Silted sand of uniform and firm character in a river bed secure from scour, and at depth below 25 feet, ...	3·50 to 4·00
Solid clay mixed with very fine sand, ...	4·00
NOTE.—Equal drainage and condition is especially necessary in the case of clays, as moisture may reduce them from their greatest to their least bearing capacity. When found equally and thoroughly mixed with sand and gravel, their supporting power is usually increased	
Sound yellow clay containing only the normal quantity of water, ...	4·00 to 6·00
Solid blue clay, marl and indurated marl, and firm boulder gravel and sand, ...	5·00 to 8·00
Soft chalk, impure and argillaceous, ...	1·00 to 1·50
Hard white chalk, ...	2·50 to 4·00
Ordinary superficial sand beds, ...	2·50 to 4·00
Firm sand in estuaries, bays, &c., ...	4·50 to 5·00

NOTE.—The Dutch Engineers consider the safe load upon clean firm sand as 5¼ tons per square foot

Very firm, compact sand foundations at a considerable depth, not less than 20 feet, and compact sandy gravel, ..	6·00 to 7·00

NOTE.—The sustaining power of sand increases as it approaches a homogeneous gravelly state

Firm shale, protected from the weather, and clean gravel, ..	6·00 to 8·00
Compact gravel, ...	7·00 to 9·00

NOTE.—The relative bearing powers of gravel may be thus described —1. Compact gravel. 2. Clean gravel. 3. Sandy gravel. 4. Clayey or loamy gravel. Sound, clean, homogeneous Thames gravel has been weighted with 14 tons per square foot at a depth of only 3 to 5 feet below the surface, and presented no indication of failure. This gravel was similar to that of a clean pebbly beach.

319. From the data in this Table it is easy to determine the requisite breadth of the concrete bed, by equating the weight per lineal foot of the wall and its maximum load with the proposed breadth of foundation, multiplied by the safe load on the soil in question. To ascertain the *thickness* of the concrete is, however, not so easy, because there are certain unknown quantities in every case which introduce an element of uncertainty. These unknown quantities are the materials of which the wall is built, the bonding, the height of the footings, the quality of the concrete. The stress on the concrete bed is not simply a compressive one, but is more or less transverse, varying with the supporting power of the ground, and the projection of the concrete base beyond the footings. On a rocky soil there would be no transverse stress, on a soil of varying density the stress would not be uniform, but for all practical purposes we may assume that the stress is uniform all along the transverse section, and that the concrete bed is in the position of an inverted cantilever uniformly loaded.

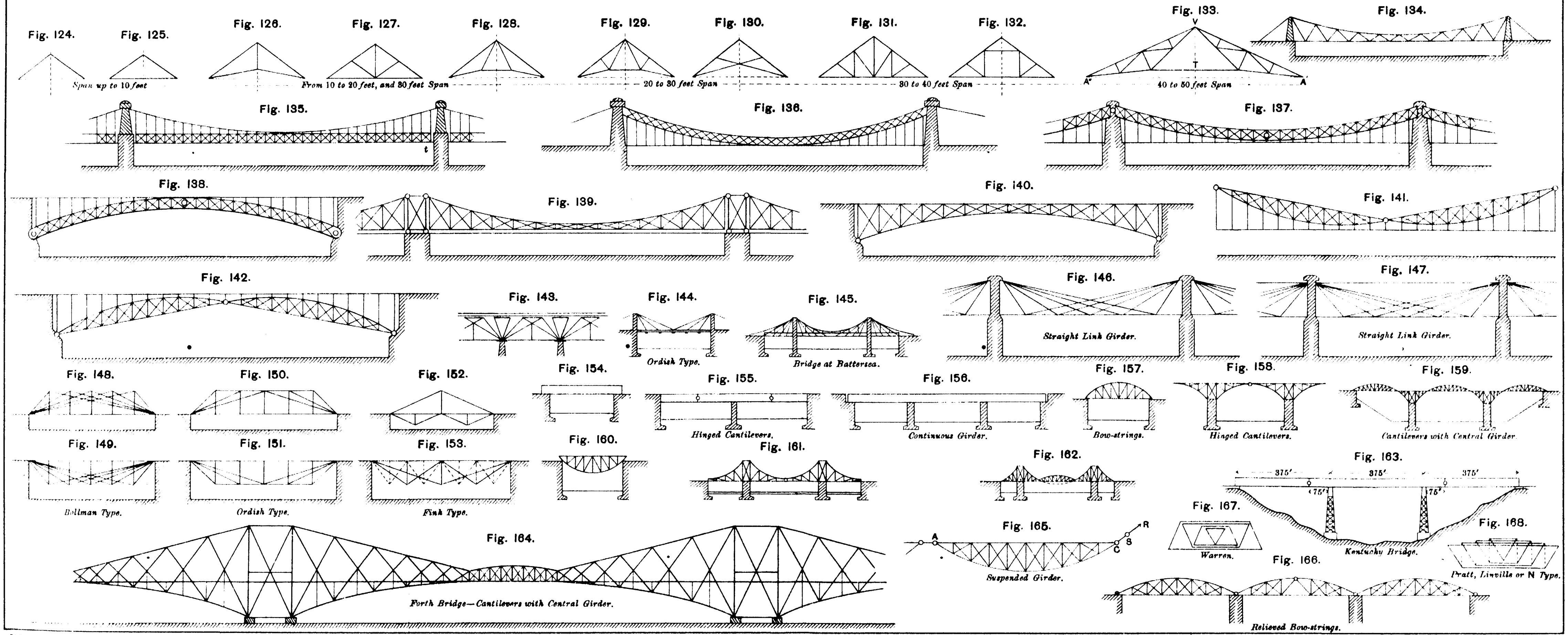
Now the ordinary formula for a uniformly loaded cantilever of rectangular section is

$$\frac{Wl}{2} = \frac{sbd^2}{6}, \text{ or } W = \frac{1}{3}\frac{sbd^2}{l}$$

where W is the total distributed load, and s = the constant, representing the extreme limit of resistance per square inch of the extreme layer of the beam. In the following example W and s are both in cwt.

320. From Table B, which gives the results of various experiments with Portland cement concrete, the value of this constant s is given in the last column but one.

STRUCTURES WHICH SPAN AN INTERVAL.



Litho., T. C. Press, Roorkee.

N.B.—It should be observed that the pairs of designs shown in Figs. 137 and 138, 139 and 140, 148 and 149, 150 and 151, 157 and 160, 158 and 161, 159 and 162, are severally the inverse of one another.

321. Take for example a three-storied warehouse with brick walls, built on loamy soil bearing 1·5 tons (30 cwt.) per square foot. Let the weight per lineal foot on the foundations, including an assumed thickness of concrete, be 130 cwt. The thickness of the wall of lower storey will be 27 inches. It is required to ascertain thickness of the concrete.

As the soil bears 30 cwt. per square foot the width of the foundations will be $\frac{130}{30}$ = 52 inches. It may be assumed that the weight on the walls is transmitted to the concrete by footings, as shown in *Fig.* 182, leaving 6 inches clear on either side. Now *if the brickwork is thoroughly well bonded and constructed* the portion of concrete AB under the footings will be subject to compression only. The projections AD, BC, will be inverted cantilevers, bearing a uniform load upward from the reaction of the earth = 30 cwt per square foot, and tending to break the concrete at A and B.

To find the thickness d:—

$W = \frac{1}{3}\frac{sbd^2}{l}$ where $W = \frac{6'' \times 30 \text{ cwt.}}{12''} = 15$ cwt., $b = 12''$

$l = 6''$, and s for concrete of 1 cement, 1 sand, and 8 shingle = 1·18

$$\therefore d = \frac{\sqrt{15 \times 6 \times 3}}{\sqrt{1{\cdot}18 \times 12}} = 4{\cdot}36''$$

which is the least thickness admissible. A factor of safety of 4 should be taken, giving the concrete a thickness of 18 inches.

322. If the weight of the wall had not been distributed by the footings, the thickness of the concrete would have been by similar calculation (with factor of safety of 4) 36 inches or 3 feet. Hence it would be a question of economy whether it would be cheaper to give a block of concrete 3 feet thick and no footings, or have 18 inches of concrete with footings. An intermediate thickness might be given were there any doubt as to the bonding or workmanship of the footings.

323. Another consideration in connection with the thickness of concrete is where the resultant pressure on the foundation does not pass through the centre, as in the case of retaining walls, abutments of arches, &c. In order to bring the line of resultant pressure within the limits of the centre third of the base it may be advisable, and useful, to increase the thickness of the concrete, as tending to lower the centre of gravity of the mass. But it must be borne in mind that the further down the foundation bed goes, the greater will be the superincumbent weight on the lowest layer, and on the earth beneath.

324. Table C gives the results of most recent experiments on the compressive strength of concrete.

TABLE C.

Compressive Strength of Concrete in Tons per Square Foot.

No	Limes and Cements	Weight per bushel	Proportion of Lime or Cement to Gravel and Sand			
			1 to 6	1 to 8	1 to 10	1 to 12
		Lbs	Tons	Tons.	Tons.	Tons.
1	Grey Lime, ...	...	10 2	4·6	5·2	...
2	Grey Lime Selenitic, ...	..	18 5	7 6	8·1	...
3	Lias Lime, ...	.	11 4	1·11	11·5	...
4	Lias Lime Selenitic, ...	..	17·2	19 6	10·2	...
5	Lias Lime, ...	..	23 0	10·7	8·5	...
6	Selenitic Lime, ...	...	26 6	15·3	13·5	...
7	Selenitic Rugby Lias,	...	37·1	34 2	21·1	...
8	Selenitic Aberthaw Lime, ...	..	34·1	21·8	15·4	...
9	Rugby Lias Cement, ...	74	17 2	10 7	5·8	..
10	Portland Cement, .	114	100·7	76 4	53·5	37·1
..	Portland Cement, ...	120	86·4	91 7	52·2	29·1

325. One of the most notable instances of the use of concrete in a restricted site for the foundations of important buildings, was in the construction of the Army Head-Quarter Offices at Simla, in 1882-84, under the superintendence and from the designs of Mr. H. Irwin, C.I.E., M. Inst. C.E., and Mr. J.M. Campion, M. Inst. C.E. The site is a very narrow one, on a very steep hill side, and on treacherous shaly soil. The buildings are three stories and four stories in height, and of somewhat irregular plan. The superstructure is composed of cast-iron columns supporting rolled beams, the whole being encased in concrete. (A very interesting account of the whole construction is contained in Proc. Inst. C.E., Vol. LXXXIII., 1885).

The quality of the material composing the concrete however, was vastly inferior to any of the concrete shown in Table C. It was all obtained locally, from Jutogh, distant four miles from Simla. The lime

was a poor lime; the surkhi was obtained from the loam overlying the limestone; the aggregate was the broken limestone. The proportions of the concrete were 1 part lime, 2 parts surkhi to 6 parts of stone broken to 1 inch gauge The concrete was laid in layers of 3 inches, and rammed until the mortar came to the surface.

It is noteworthy that much of this important work was done by unskilled military labour.

326. In America, layers of steel rails are sometimes used in conjunction with concrete in foundations of lofty structures. A bed of concrete is first laid, then a layer of rails close together embedded in fine concrete, then another layer of rails at right angles to the first, then a course of 6 inches stone, then the superstructure.

327. When the weights of the different parts of a building (*e.g.*, the tower and main body of a church) are very unequal, they should be disconnected both in the footings and above ground, no bonding should be permitted between the two, the joint should be closed by a chase, and even friction should not be allowed to act between the parts. After the building has settled, the line of connection can be closed up.

328. Where, owing to the slope of the ground, the bottom of the excavation is cut in steps or *benches*, the foundations should be brought up to a uniform platform for footings in concrete, or in brickwork in cement. If they be built in mortar the greater number of joints in the deeper part will cause unequal settlement and cracks.

329. Where the ground is unequal, and the softer portions cannot be arched over, the foundations at those portions must be made wider and thicker, in accordance with the principles described above.

330. It may be noted that sand is an excellent foundation provided it be confined so as to be free from lateral movement, and from any chance of under-scour. If there is no possibility of under-scouring, or of the sand being drawn off by pumping operations, a load of from 5 to 6 tons per square foot may be imposed with safety.

331. In parts of the delta of many Indian rivers, the soil, though apparently firm near the surface, grows looser the deeper the excavation proceeds, ending possibly in a quicksand. It will often be found best in such cases to put in a mass of concrete in cement over the whole area and build on it, and thus practically float the structure on the treacherous soil.

SECTION III.

CHAPTER XXII.

THE DESIGN OF STRUCTURES THAT SPAN AN INTERVAL.

332. In Chapters I. and II of this Volume the general question of the Stability of Structures is considered, and in paras. 18 and 24 it is shown that those which span an interval may be divided into three general classes according as the pieces immediately supporting the load or loads are subjected to tensile, compressive, or combined tensile-compressive strain, the corresponding characteristic structures being the suspension, arch, and parallel-flanged bridge, respectively. It is shown that these three classes embrace all structures whatsoever that span an interval, whether they be beams or trusses of roofs, carrying a stationary or dead load only or its equivalent, or floor joists and the several types of bridges, whether of iron, wood, or masonry, which are subjected to the simultaneous action of a moving, or live, as well as of a stationary or dead, load. Now, the figure of a Roof or Bridge-frame is generally closed; that of a Suspension or Arched structure open; but in the Chapters referred to it is shown that by supposing a weightless bar to extend across the interval at the springing level of an arch, or a tie-rod to join the points of attachment of the suspension chain or braced rib, the figures formed by the resultant lines of resistance of both these latter structures become closed; so that all three classes may, in fact, be dealt with under a heading in which the bending moments are resisted by couples which are fully determinate. So much for the stability of these structures. With regard to their design a few brief general remarks are added.

333. The covering of the iron roof, or roadway of the bridge as the

case may be, is carried on a frame, or truss, to which it is attached in the former case by purlins, in the latter by cross girders; and the structure must be designed so as to be strong enough to carry its own weight and imposed load, as well as resist the action of wind unsymmetrically applied. But while in the roof-frame the external load is, as a rule, applied along enclosing pieces of the structure which are inclined to the horizon, in the bridge-frame it is applied along horizontal, or almost horizontal, ones; and while a roof-frame, if inverted, has, as a rule, no practical value, a good bridge-frame affords, when inverted (as we have seen) a new useful design.*

With a view to assisting the Student, we shall quote some remarks, from the best authorities, on the subjects of the *loads on bridges*, their *cross girders*, and *stiffening arrangements*.

334. Whatever the special function of the structure be, its pieces, as has been already repeatedly pointed out, must be so designed as to be capable of resisting with safety the greatest possible bending moments and shearing forces to the action of which it is ever likely to be exposed. In hinged and jointed structures, as will be more fully explained later on, the position and amount of the moving load producing this greatest effect in any given piece may often be conveniently determined by applying Ritter's Method of Sections, already referred to in para. 259 of Vol. I., the load being apportioned at certain joints and the structure supposed divided by a plane cutting three pieces only; it may then be ascertained whether any particular load produces tension or compression in the bar under consideration by observing in which direction such load tends to make that part of the structure rotate. In this manner the joints that must be loaded in order to produce tension or compression in the selected piece can be ascertained, and the maximum stress found by loading all the former or all the latter joints as the case may be. For a preliminary investigation the weight of the bridge may be calculated by any of the formulæ given in Chapter XIV., Vol. I., or by Sir B. Baker's Tables, given in the Appendix to this Volume, or by Claxton Fidler's formulæ given in Chapter VII. of his *Practical Treatise on Bridge Construction.*

335. The relative importance of the two classes of load above referred to (*i.e.*, dead and live load) depends on the length of span, and in

* Bow's *Economics of Construction*, p. 9.

regard to this Sir B. Baker places Railway Bridges spanning intervals of from 10 to 300 feet under the head of Short Span Bridges, and those including spans of from 300 to 3,200 feet under that of Long Span Bridges, and this classification he justifies in the following words* :—

336. "In a long span bridge, so defined, the weight of the structure itself will constitute at all times a very important, and at the higher spans, an all-important, proportion of the gross load; and since the action of a rolling load is very different to that of a dead load of equal intensity, the general form of bridge suitable to a mixed load when the rolling element is of insignificant proportion may be essentially different from the design suitable for a bridge when those conditions are reversed.

"Again, a bridge may be defined as of long or short span relative to the average length of train traversing it; because a train does not constitute a rolling load of uniform weight per lineal foot, and the variation in its density will be of little importance in the instance of long spans, but of vital influence in the case of short span bridges. Thus, a passenger train may be made up of a string of vehicles weighing less than half a ton per foot run, whilst for the 50 feet length at the head of the train, occupied by the engine and tender, the weight may average $1\frac{1}{4}$ tons per foot, or two-and-a-half times as much as the remainder of the train; and in the very common case of the employment of two engines and tenders the heavier load will apply of course to a length of 100 feet of the train. In the same way a goods train may consist of a long line of trucks weighing less than $\frac{3}{4}$ ton per lineal foot, whilst the tank engine at its head may average double that amount, or $1\frac{1}{2}$ tons per foot. Hence, as the wheel base of a tank engine is not unfrequently as little as one-half of the total length of engine, it is obvious at once that in short span bridges we shall have to entertain rolling loads at least as high as 3 tons per lineal foot, although in the case of long span bridges, subject to be traversed by the same trains, the average weight of the train, or little more than one-fourth of the preceding amount, need be considered in the calculations.

"It will, we think, be granted at once, that bridges which in the ordinary working of the traffic may be loaded throughout their entire length with engines and tenders—that is to say, bridges up to 100 feet in span—are practically, in many essential features, short span bridges. If due

* *Long Span Bridges*—Sir B. Baker, p. 87, *et seq.*

consideration be given to the influence of the heavy load at the head of the train in traversing a long-span bridge, it will be admitted also that the 300 feet span, adopted as the commencement of the long-span series of uniform rolling loads, has not been fixed at too high a limit."

337. The same distinguished author summarizes as follows "the special action of a rolling load upon a girder as compared with a dead load of equal intensity"*:—

"(1). The rolling load may impose a greater bending stress than that due to the same load in a state of rest, and this would be accompanied of course by an increased unit strain upon the metal and an increased deflection of the girder.

"(2). The repeated applications and withdrawals of the load with the consequent bendings of the girder may 'fatigue' the metal, and a lower unit strain may, on that account be desirable.

"(3). In the case of a railway bridge, with permanent way inefficiently maintained, these repeated applications of the load will be accompanied with vibrations which may have the effect of multiplying the number of bendings of the girder to an indefinite extent, and so increase the 'fatigue' upon the metal."

338. *The Engineering Record* of 5th May, 1894, states that similar results have been arrived at by Professor Mélan, who, in a paper recently published in the *Zeitschrift*, gives an analytical investigation of the effect of moving loads on metallic bridges.

The conclusion is reached that for railroad bridges the actual rolling load should be increased by the following percentages† to bring it to an equivalent statical load:—

Span in feet	6·5	13·1	16·9	32·8	49·2	65·6	98·4	131 2	262 5	393·7
Percentage increase, ..	80	71	67	54	44	41	34	30	23	20

These remarks should be compared with those already made in the

* *Long Span Bridges*—Sir B. Baker, p. 80.

† These percentages are derived from the formula—

$$\text{Percentage increase} = 14 + \frac{2600}{L + 23} \text{ (in which } L = \text{span in feet).}$$

Addendum to Chapter XXI., Vol. I., the Table of Effective Loads given on page 5 of which applies, as therein stated, to English stock running over the line at the time the Table was constructed, and Sir B. Baker's remark that "the precise loads that should be provided for in any particular line of railway is, of course, a matter for the judgment of the Engineer," all important. The loads given, since they apply to narrow (4′ 8½″) gauge stock only, would have to be increased, for instance, for the Indian broad gauge (5′ 6″) lines of railway, and correspondingly diminished for metre gauge lines. And it should further be borne in mind that "the heaviest types of engines now used abroad weigh as much as 101 tons, with a total wheel base of 54 feet 3 inches. The greatest load on a 10 feet span would be 51 tons (eight-wheel coupled engines, 4·5 feet centres, and 40,000 lbs. on each axle), thus giving 5 tons per foot run for the load. In estimating, therefore, the probable rolling loads on broad gauge mountain railways, with ruling gradient of, say, 1 in 40, it would be advisable to take such heavy engines into account." *

The Cross Girders. †

339. "Probably the changes which have been effected in modern bridge construction are not more forcibly evidenced by any single instance than by that of the differences as regards the strength and general arrangement of the cross girders. In early railway bridges the cross girders were, apparently, considered as subject only to the same load per square foot of the platform as the main girders, and the case of the concentrated weight of a pair of driving wheels, which must come with its full intensity upon each cross girder, however close they may be spaced, was not entertained; nor indeed was it so essential to do so then, when the difference between the most heavily and lightly loaded wheels was far less marked than at present.

"The distance apart of cross girders is governed to some extent by the position of the stiffeners in the main girders, even when of plate construction, but in the instance of lattice girders the proportions of the triangulations obviously fix the position of the cross girders in a far more arbitrary manner. In ordinary cases it may be considered as generally advantageous to space the cross girders at distances equal to about

* Paper by Major Scott Moncrieff, R.E., on *Rolling Loads*, p. 3

† *Long Span Railway Bridges*, p. 119.

$1\frac{1}{2}$ times the wheel base of two pairs of coupled driving wheels, or say from 9 to 12 feet, and to place rail girders between the cross girders under each rail. In the large bridge over the Danube, near Vienna, the cross girders are 2 feet 7 inches in depth, and 12 feet 6 inches apart, and the rail girders are 1 foot 4 inches deep. If rail girders or their equivalent in the form of timber beams be omitted the cross girders may be placed 5 feet apart, and there can, in no ordinary case, be any justification for placing them closer together, since an ordinary bridge rail and longitudinal sleeper, properly qualified to take the traffic of the railway when laid in ballast, will also be strong enough to bridge the opening between cross girders so spaced.

"It is true that even at the present time the spacing of cross girders at 2 feet intervals may be justifiable if care be taken to connect the whole together by stiff distributing girders, so that successive deflections of each cross girder may be superseded by a general deflection of the whole platform for at least the entire length of the wheel base of the engine. But even then, as a single girder will carry a given load with a less weight of metal than two girders, unless the load be excessive, it follows that the close arrangement can only be desirable in instances where the cross girders are necessarily very shallow, and where, to get in the necessary sectional areas, they would require to be of box construction if spaced far apart.

"Considerable economy will be found to result from giving a liberal depth to the cross girders. From $\frac{1}{8}$th to $\frac{1}{7}$th of the span is not an unusual proportion for the depth in modern bridges.

"The weight of the bracing will have to be considered in connection with that of the cross girders." "In old bridges bracing was frequently omitted, and the lateral stability was dependent upon the rigid attachment of the cross girders to the main girders, hence great stress at those points would occur during a gale. Continental Engineers, in proportioning the strength of the horizontal bracing of their bridges, very usually take the force of the wind as limited only by the power of an ordinary passenger train to resist overturning, for they reasonably argue that if the train be blown over it cannot run over their bridge.

"The weight of the cross girders and bracing for a railway bridge to carry two lines of railway between the main girders may be taken as an average to vary from 6·7 cwt. for a 20 foot span to 9 cwt. for a 275

foot span; but it will be understood that considerable modification in these weights, both of a plus and minus nature, may be effected by a variation in the depth or arrangement of the girders."

Horizontal and Vertical Bracing of Bridges.

340. It has been already explained that open ironwork structures require bracing in a vertical plane in order to counteract the tendency to distortion produced by variations in the load acting in that plane. Large, deep bridges, however, require to be braced not only in the vertical, but in the horizontal, plane also, because * "the rolling load in passing over a bridge does not produce a simple vertical pressure on the bridge. In the case of a railway train, the inequalities of the way set up both horizontal and vertical oscillations of each carriage, which modify the vertical load and introduce horizontal pressures on the rails. The primary object of the horizontal braces is to resist these horizontal forces, by forming with the booms of the bridge, and the platform, a horizontally-placed girder of a depth equal to the width of the bridge. And the primary object of the vertical bracing is to bind together the two main girders, so as to present a combined resistance to the torsion and tendency to overturning produced by the horizontal forces.

"Whether the platform is on top or at bottom of the main girders, it is obvious that the horizontal bracing will be most effective when the rails, the booms, and the bracing are as nearly as possible in one plane.

"When there are no cross beams, the horizontal bracing bars are alternately struts and ties, according to the unknown direction of the horizontal forces. In such cases the horizontal bracing should consist of tee-iron bars, placed back to back, and rivetted to the flanges of the main girders. Tee-irons, $4'' \times 3'' \times \frac{3}{8}''$ to $4'' \times 4'' \times \frac{1}{2}''$, intersecting the axis of the bridge at angles of 30° to 40° are sufficiently strong in ordinary cases. When there is a platform with cross beams, the cross beams may be utilized as struts in the horizontal bracing, and the bracing bars may be flat or round bars, rivetted to the main girders and to the cross beams, where they intersect them. These bars may be 6 inches to 8 inches wide, and $\frac{3}{8}$-inch to $\frac{1}{2}$-inch thick.

"The magnitude of the horizontal force developed by the lateral oscillation of a train is, I believe, unknown. The wind pressure alone

* *The Construction of Wrought-iron Bridges*, Lecture III., February 15th, 1871, by Professor Unwin.

acting on the side of a train of carriages might produce a pressure of perhaps 3 cwt. per foot run,* or say a horizontal force of $\frac{1}{7}$th the rolling load. So that we shall, I believe, make no excessive allowance if we suppose, in single line bridges, that the horizontal force acting on the rails may reach $\frac{1}{5}$th of the rolling load. When the platform is on top of the main girders, this force tends to overturn the girders, and must be resisted by building the girders into the masonry of the piers, or we may introduce over each pier a frame of vertical bracing, binding together all the girders and spreading over a wide space the resistance to overturning. Very considerable differences will be found in existing bridges as to the amount of this vertical bracing, but it seems reasonable that the frame, at each end of a span of a single line bridge, should be at least strong enough to resist a horizontal force of $\frac{1}{10}$th the rolling load on the span, applied in the plane of the rails. And the resultant of this force, and the half dead and live load of the span, should of course fall within the frame. If the bridge has several spans, so that the girders of two spans are united on one vertical bracing frame over the pier, that frame requires double the strength of frame previously considered. When a bridge carries two lines of way, the effect of wind pressure bears a less proportion to the rolling load, but I am hardly prepared to recommend a less strength in the bracing. Of course the suggestions made here are to be taken only as general guides. The exact nature and amount of the horizontal straining forces are unknown.

"With girders of moderate span no other bracing frames than those over the piers are necessary. But many Engineers introduce vertical bracing frames at distances of about 20 to 30 feet along the span.

"With the platform on top of the main girders the horizontal bracing has the subsidiary effect of stiffening the compression booms against lateral flexure. When the platform is below the main girders, the top booms can have no aid from bracing unless the depth of the main girders is 15 feet, and then light top bracing is introduced to stiffen the booms. With the platform at bottom, the vertical bracing is dispensed with."

* This is in England.

CHAPTER XXIII.

FRAMED STRUCTURES.

341. As all the Iron and Steel structures coming under consideration, including arch and suspension structures, may be regarded as framed structures, it will be well, before proceeding further, to offer a few remarks regarding them.

342. In *Plate* XXXI. a few frames are shown, but no attempt has been made to classify them; for this, Students are referred to *Bow's Economics of Construction*, in which little volume no less than 337 frames are classified, and stress diagrams drawn for between 140 and 150.

343. All frames, forming the supporting or carrying members of a structure, which spans an interval, are divided into an arbitrary number of equal parts or *panels*, and the load imposed at each division, or panel point, is supported, or carried (as the case may be), by some piece or pieces meeting at that point.

344. Those framed structures which are composed of separate systems, each carrying its own portion of the load, are distinguished as *Composite Structures;* those, on the other hand, in which two or more systems offer their combined support to the same element of the load, are defined as *Combined Structures.* The former class includes frames composed of parallel systems, such, for instance, as Lattice girders, as well as those consisting of a number of systems superposed on one another, examples of which are shown in *Plate* XXXI. Among members of the second class it is often difficult to decide which of the several systems actually bears the load, or, rather, what proportion of the load should be allotted to each, and the question can often only be decided by reference to the laws of elasticity.

Of Trussing there are two kinds—*Simple* and *Multiple.*

I. *Simple Trussing.*

345. (1). *Triangular Frames.*—If the load at each panel point be supported by a separate triangular truss, extending the whole length of the span, the truss becomes in principle the *Straight-link-girder*, *Figs.* 146 to 149. This frame is seldom employed in the upright position, *Fig.* 148, but in the inverted form it represents the American *Bollman Truss*, *Fig.* 149, and if the boom shown in *Fig.* 147 be removed, the frame is transformed into the *Straight-link-Suspension-Bridge*, *Fig.* 146. The load borne at the apex of each triangle being known, the stresses in the pieces can be determined either by diagram or calculation.

(2). *Trapezoidal Frames.*—Instead of a series of triangles, the frame may be composed of a series of trapezoids, as shown in *Figs.* 144, 145, 150 and 151; when the boom shown in *Figs.* 150, 151 is removed, the figure represents in principle the *Ordish Suspension Bridge*, *Fig.* 144. The bridge at Battersea, *Fig.* 145, consists of a suspension bridge, stiffened by a plate girder and *Ordish shrouds.*

II. *Multiple Trussing.*

346. Instead of placing a complete triangular or trapezoidal system at each panel point, the simple systems may be superposed on, or placed within, one another; thus triangles may be placed within triangles, as so frequently occurs in Roof trusses, or trapezoids placed within trapezoids, or superposed on triangles.

347. Thus, in the common King-post truss, *Fig.* 127, we have four smaller triangles placed within a large one; in the frame shown in *Fig.* 129, five smaller triangles are placed within a large one, and so on. In girders of the *Fink* type, *Figs.* 152 and 153, a complete triangular system is first placed extending over the whole span, then a secondary and similar system, extending over each half span, then a tertiary system extending over each quarter span, and so on.

348. Again, an inverted triangular system might be placed with its apex at the central division of the span and its base extending the length of two panels; this triangle might then be enclosed within an inverted trapezoid, whose lower horizontal side is equal in length to the base of the triangle and upper side to twice that length; this trapezoid and triangle might then be placed within a second inverted trapezoid, whose lower horizontal side is equal in length to the upper side of the

previous trapezoid and upper side longer than that length by the length of the base of the inner triangle, or two panels width, and so on. If now the joints of the frame so formed be connected by vertical posts, the form will represent in principle the *Pratt*, *Linville* or N type of girder,* *Fig.* 168; if, on the other hand, there be no vertical posts, and the trapezoids be alternately upright and inverted, the frame assumes the form of a *Warren* girder, *Fig.* 167.

For further information on this subject, Students are referred to Chapter VI. of *A Practical Treatise on Bridge Construction* (1887), by Claxton Fidler, from which much of the preceding information has been taken.

Hinges in Girders.

349. It is found convenient to hinge, or pin together, certain portions of framed Metal Bridges, and the purposes for which this arrangement is employed are two in number, *viz.*:—

(*a*). To meet strains due to variations of temperature.

(*b*). To locate points of contra-flexure.

350. With regard to (*a*), it is evident that a metal rod, if freely exposed to climatic variations of temperature *must* suffer considerable expansion and contraction of figure, and it, therefore, becomes necessary, when designing metal structures of considerable size, to so arrange that such tendency to deformation shall have as free play as possible, and in no way interfere prejudicially with the state of internal stress which is necessarily set up in the pieces by those loads which it is their proper function to carry.

351. In frames provided with straight lower booms, such as parallel girders, upright bow-string girders, roof trusses, &c., strains of this nature are sufficiently provided for by placing the ends of the boom on rollers, as has been already explained in paras. 392 to 394, Vol. I. A continuous girder might, for instance, be fixed to one pier and allowed to rest on rollers at all the other points of support. But such an arrangement would be neither sufficient nor suitable for *all* forms of girders and under all circumstances. It would be unsuitable, for instance, for the parallel stiffening girder which has to meet the effects of

* Girders in which the web-bracing consists of vertical posts and inclined braces, are known in America under various names, such as the *Linville*, *Pratt*, *Murphy*, or *Whipple-Murphy* trusses, according to some variations of detail which are not always to be easily distinguished (*Practical Treatise on Bridge Construction*, p. [illegible], foot-note.)

variation in the length of the suspension chain to which it is attached, because the stiffening girder must be prevented from rising at each extremity; nor could this arrangement be carried out for the braced suspension or arched rib which replaces the suspension chain and girder. It will, however, be seen that all cases of strain such as we are considering are sufficiently provided for by simply *hinging, or pinning* together, the two portions of the bridge girder or rib, because then the joint is free to rise or fall with every change of temperature.

352. In arch or suspension structures constructed in rigid form, sufficient allowance for temperature strain is afforded by placing a hinge either at the centre of the span, or near one of the abutments; or hinges might be placed at midspan as well as at each abutment.

353. With regard to (*b*) it has been shown that the positions of the points of inflexion or contra-flexure of a continuous girder depend on the loading, the girder's continuity and the resulting conditions affecting its curve of deflection, so that it becomes necessary for a fresh diagram to be drawn, or calculation made, for each new position of the load, while the results are, to a certain extent, ambiguous, and rest on assumptions which seldom correspond with the actual conditions of the case. But it will be seen at once that if the continuity of the girder be broken by severing it at the point of contra-flexure and introducing thereat a form of connection practically amounting to a hinge, the points of contra-flexure will thereby become mechanically fixed, and that this advantage will be obtained without sacrificing in any way the stability of the structure, provided only that the hinges be so located as to lead to no distortion of the whole structure.

354. These hinges would obviously be placed, as a rule, on either side of a central pier—in the case of a three-bay bridge, in a side span.

355. Suppose, for instance, *Fig.* 41, *Plate* IX., to represent in outline a continuous girder, supported at the points A_1 and A_4, and supported and anchored down at the points A_2 and A_3. It is evident that, if the girder be divided and hinged at the points y_2, x_2, y_3 and x_3, these points must, of necessity become points of inflexion; that as the load varies both in amount and position, the resultant lines of resistance will so adjust themselves as to constitute these points, points of inflexion; and that the structure will, under all circumstances, consist of the three independent girders, A_1y_2, x_2y_3, and x_3A_4, supported at the

points A_1 and A_4 and at the extremities of the double cantilevers y_2x_2 and y_3x_3. This method of construction is followed in all the examples of stiffened Suspension and Cantilever bridges, shown on *Plate* XXXI., and it is obvious that the introduction of the hinges can be attended with no possible distortion of the structure under any position of the load, provided only that the cantilevers be properly anchored down at points corresponding to A_2 and A_3, *Fig.* 41.

356. In the Kentucky Bridge, shown in outline in *Fig.* 163, the hinges are placed in the side spans, the centre span being continuous. The bridge consists of three equal spans of 375 feet each, each side span being hinged at a point distant 75 feet from the adjacent pier. If *Fig.* 41 be taken to represent the bridge in skeleton, then the points y_2 and x_3 would represent the positions of the hinges, the centre girder being continuous and supported at points A_2 and A_3.

357. In order to locate the points of inflexion, then, hinges may be introduced either in the central or side spans, and in each case the girder would be supported at the piers on rocking bearings (para. 391, Vol. I.) so that the parts might be capable of assuming freely any slope that their deflection curve might necessitate. Under these circumstances the stability of the structure would obviously not be affected by the introduction of the hinged bearings, and the consequent simplification introduced into the calculations is no less obvious.

358. The longitudinal expansion and contraction of the entire length of the girder for variations in temperature would be met, as before explained, by fixing the girder to one support, generally a pier, and providing rollers at all the other supporting points.*

Roof and Bridge Frames.

359. The treatment of the more common forms of Roof Frames has been fully discussed in Chapter V., Vol. I., and it is only necessary to add that the judicious selection of a point about which to take moments will often enable the stress diagrams of the more complicated frames to be drawn. Thus, for instance, a little consideration will show that a stress diagram cannot at once be drawn for the form of roof shown in *Fig.* 133, *Plate* XXXI., but if a vertical plane be supposed to divide the frame symmetrically passing through the vertex V, and one-half be

* The above information is largely taken from *Claxton Fidler's Practical Treatise on Bridge Construction*

supposed removed, by taking moments about V, it is evident that the moment of the weight of the unremoved portion of the frame is equal to that of the tension in rod T. Hence, the tension in T is known, and the diagram can be completed.

360. Nor should any difficulty be experienced, after what has been said in the earlier Chapters of this Volume, in drawing stress diagrams for Bowstring Girders, whether upright or inverted, and Bowstring Roof Frames, loaded, if necessary, simultaneously with roof covering and snow, and also with wind pressure* unsymmetrically applied to the roof, the loads being distributed in all cases at the joints of the frame only.

361. Bridge Frames may be ranged under the following twelve heads, including, as before explained, all arch and suspension structures, the limiting span being reached when the weight per foot run becomes infinite†:—

TYPES OF BRIDGES.

TABLE I. *Suspension and Arch Systems.*

Maximum Span, in feet		Order	Suspension	Arch.	Order	*Maximum Span, in feet*	
Iron	Steel					Iron	Steel
1,300	2,000	1	Chain with stiffened roadway.		1		
		2	Chain braced to roadway.		2		
		3	Braced ribs.	Braced ribs, ...	3	1,300	2,000
		4	Inverted Bowstring.	Upright Bowstring,	4	900	1,400
1,300	2,000	5	Suspended girder	...	5		
1,000	1,600	6	Straight-link suspension.	Upright Bollman and Ordish frames, ...	6		
		7		Masonry arch, ..	7		

* In the present incomplete state of opinion as to the actual effect of wind pressure on inclined surfaces, it may be as well, with reference to the method described in para 116 (*b*), Vol I., and to the subsequent remarks on this subject, to sum up briefly the general conclusions which would seem fairly to express the results of the most recent experiments. They may be stated as follows, as gathered from a lecture recently delivered by Professor Unwin —

(1). The theory that the wind pressure acts on one side only of a roof, and is uniformly distributed normally over that side, appears to be erroneous

(2). Up to a slope of 30°, roofs which are built with continuous walls beneath them may be considered safe from wind pressure as long as the walls are safe.

(3). Sheds and other open buildings, as long as they are not a great height from the ground (say within 20 feet) are sufficiently secure against downward wind pressure.

(4) All sheds and roofs with very deep eaves and verandahs, especially of buildings with several stories, should be firmly tied down to the walls or supporting pillars, so as to be secure against the upward effect of the wind.

(5). Valleys are objectionable as sites for buildings owing to their liability to ricochet wind force and to snow accumulations

† Sir B. Baker's *Long Span Railway Bridges*

TABLE II. *Parallel and Continuous Girders.*

Maximum Span, in feet. Iron.	Order.	Girders supported or fixed at the extremities, and also intermediately supported	Maximum Span, in feet. Steel.
700	8	Box plate girders, ...	...
700	9	Lattice, Warren, Whipple-Murphy, &c., ...	1,000
700	10	Straight-link girders, with booms, ...	1,100
1,200	11A	Cantilevers, with parallel booms, ...	1,700
1,400	11B	" of varying depth, ...	2,000
	12A	Continuous girders, with parallel booms, ...	. .
2,300	12B	" " of varying depth, ...	4,000

362. Sir B. Baker, in *Long Span Railway Bridges*, estimates the average weight of iron per foot run required for each of the above types of bridge, and shows the results collectively in a diagram, "the curved lines of which are obtained by plotting the gross weight of iron in cwts. per foot span * * * to the vertical scale of 100 cwt. to the inch," the span being shown on a horizontal scale of 300 feet to 1 inch. "A glance at the diagram will be sufficient" he says "to assure us, that the several types do not maintain the same relative economic position throughout, since in many instances the lines cross one another, showing that, at the span corresponding to the point of intersection on the diagram, the weight of metal required in the construction of a bridge on either of the systems in question will be identical."

363. The arrangement of types adopted by Sir B. Baker is different to that given in the above tables, so the type-number is omitted in the following extract from p. 79 of his *Long Span Railway Bridges*:—

"Briefly summarizing the results exhibited in the diagram for iron structures," which, however, be it observed, only indicates the comparative merits of the several types *in so far as the superstructure of the main span* is concerned," we find that at the span of 300 feet the straight-link suspension* obtains an advantage of some 20 per cent. over any other system, and that it maintains a certain advantage of diminishing value up to 700 feet span, when it has to resign the lead to the continuous girder † of varying depth, which type maintains a rapidly increasing advantage over all others up to the limiting span. These

* *Fig.* 146, *Plate* XXXI. † *Figs.* 159 and 162

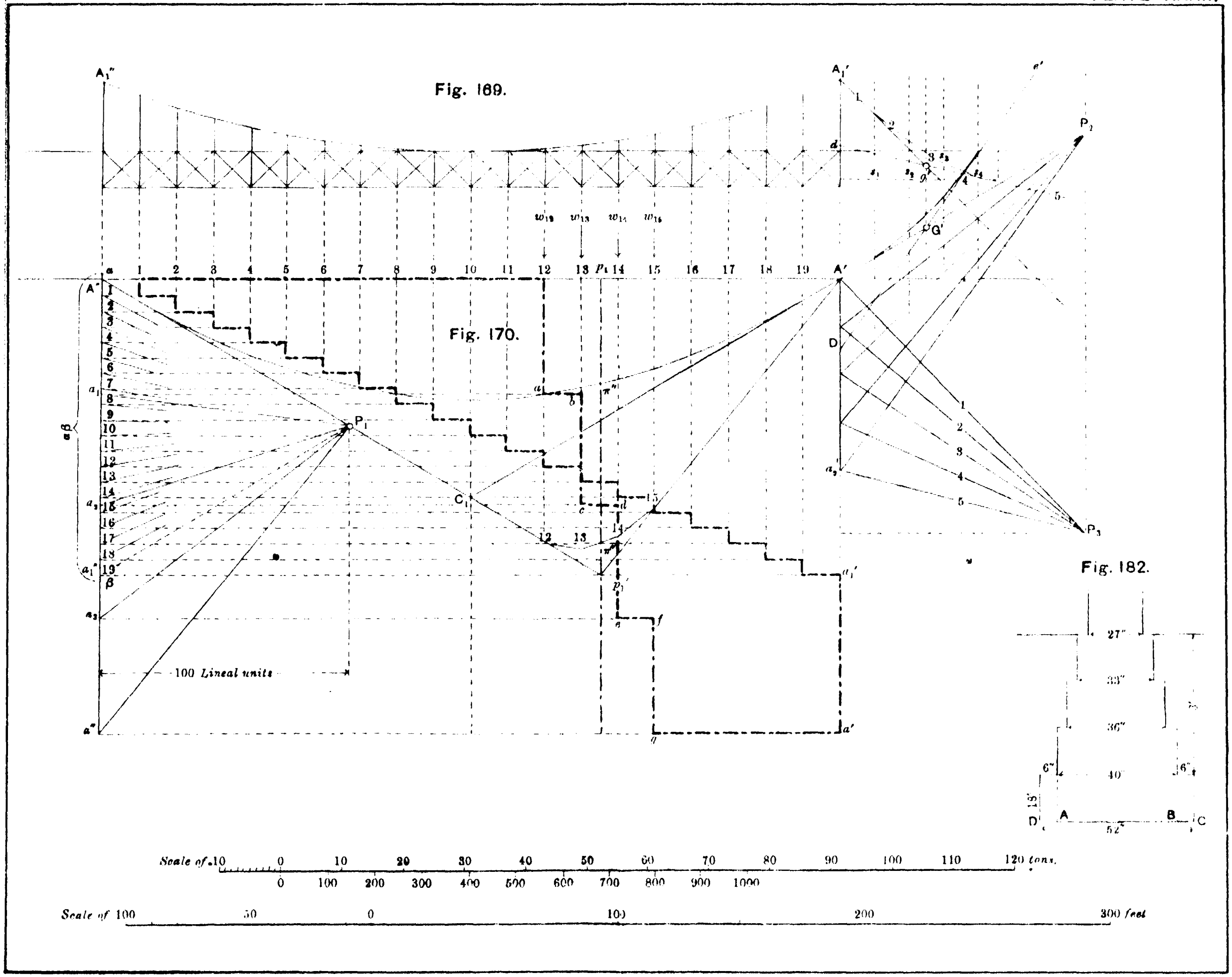
two forms of construction, then, within their own proper spheres, appear to be the most economical possible, as regards the superstructure of the main span. It is obviously quite possible that in many instances anchorage could not be obtained for the suspension bridge except at a cost which would render even our heaviest type, the box girder, a more economical form of construction.

364. "The system ranking second in the scale of economy is the cantilever lattice girder of varying depth, which maintains its relative position throughout, unaffected by the specific length of span. The suspension with stiffening girder* and the suspended girder† succeed the last-named one. Although palpably different both in principle and appearance, the respective weights are almost identical throughout, being, up to 700 feet span, little different to the preceding type. We now come to the cantilever lattice girder of uniform depth following closely on the heels of the last two systems up to 600 feet span, when it is superseded by the arched rib with braced spandrils.‡

365. "The independent girders, as might be fairly expected, occupy the lowest place on the list, although at 300 feet span the straight-link girder shows a slight advantage over the arch. Within the limits of 400 or 500 feet span the straight-link is the most economic form of the independent girder; above that span the bowstring girder surpasses it. The lattice and box girders conclude the list.

366. "We have already insisted that the order of economy thus exhibited holds good only with reference to the superstructure of the main span, and that it does not represent the comparative costs of complete structures on the different systems, for which, indeed, no general rule can possibly be given. Some of our types require loftier, and, *cæteris paribus*, more expensive piers than others; thus the piers for a suspension bridge will be higher than those for an arch, and the land-chains required in the former system will be another cause of excess. Probably the fairest way of arriving approximately at the true relative economy of the different systems in ordinary cases will be by ascertaining the gross *average* weight of iron per foot run required in the construction of viaducts consisting of three spans, of which the side spans are one-half the opening of the centre one. Under these circumstances, * * * the difference of weight between the independent girders

* *Fig.* 135. † *Fig.* 165 ‡ *Fig.* 140.



Litho., T. C. Press, Roorkee.

and those dependent upon some extraneous means of support is much less marked than before. Indeed in some instances the conditions are reversed; thus, at 300 feet span both the lattice and bowstring girders show a certain positive advantage over the arch, whilst the more economical system of the straight-link retains an advantage even up to 400 feet span.

367. "The positions of the several types in the scale of economy would, obviously, be again shifted if the side spans were taken at a different ratio to the main span than $\frac{1}{2}$."

368. There are, however, many considerations besides the mere average weight of a bridge to be duly weighed before the particular form which it is most desirable the bridge should take can be decided on, and of these none is more important than those affecting *the facility of erection.* In the majority of cases the cost of scaffolding is in itself prohibitive, even were its adoption not actually prohibited by the necessity of keeping the navigable channel across which the bridge is to be thrown free from all obstruction. Under these circumstances the bridge must be either built at the nearest available spot and lifted or slid into position, or else the design must be such as to admit of the bridge being put together *in situ.*

369. The types of Iron Bridges to be specially dealt with in the following pages are:—

(1). The Stiffened Roadway, employed in conjunction with the flexible suspension chain, *vide Figs.* 135 and 145, *Plate* XXXI.

(2). The Stiffened Suspension or Ached rib, designed as a deep girder, *Figs.* 136 to 142.

(3). The Suspended Girder, *Fig.* 165.

(4). The Continuous Girder.

(5). The Hinged Cantilever, and Cantilever-girder Bridge, *Figs.* 158, 159, 161 to 164.

(6). The Masonry Arch.

CHAPTER XXIV.

SUSPENSION, ARCH AND CANTILEVER STRUCTURES OF WROUGHT-IRON AND STEEL.

370. The general method of determining the figure which a perfectly flexible chain, or lineal arch, assumes when carrying a given load across an interval, has been fully discussed in the earlier Chapters of this Volume; it has been shown that the figure depends upon the distribution of the load, and is, in fact, identical with the diagram of bending moments corresponding to the particular distribution under consideration, and although, it may be observed, the flexible arch is, in itself, inconceivable, yet we shall continue to deal with it in conjunction with the flexible chain, because, mechanically, the one is the reverse of the other, and because the linear arch being the line of resultant resistance becomes perfectly conceivable when dealing with stiffened structures.

371. Suppose a heavy chain to be suspended between two points and to be acted on by no external load; the curve of the chain would be that of the common catenary. On the other hand suppose the chain's weight to be neglected, and a uniform load to be spread over a horizontal platform, and the platform to be connected with the entire length of the chain lying between the supports by means of vertical suspenders, or rods; the figure of the chain would then be a parabola, or parabolic polygon, according as the rods were infinite in number, or only numerous. It is evident that if the weight of both chain and platform be taken into account, and act simultaneously, the figure of the chain will more nearly approximate to a parabola than a common catenary the greater the weight of the uniformly loaded platform is in comparison with that of the chain.

372. Now structures such as we are considering are seldom, if ever, employed to cover any but long spans, so that the weight of the platform and what it carries is great in comparison with that of the chain,

or other suspending or supporting arrangement, and, moreover, the flatter these curves are the more nearly do they approach one another in figure. Therefore, on the whole, we may conclude that the figure of the chain or linear arch more nearly approaches a parabola than any other geometrical figure; so much, at least, may be assumed with sufficient accuracy for all practical purposes; so that the resultant stresses arising may be dealt with on this hypothesis.

373. Were the load steady and its distribution constant and not liable to variation in position or amount or to disturbance through the action of wind, rain, or other cause, it would be sufficient to design the structure on the supposition that the resultant line of resistance approximates very closely indeed to a parabola. It would be sufficient were the structure naturally stiff as in the case of the masonry arch. But all such structures as are liable to displacement of their pieces through change of position or amount of the applied load it is necessary to stiffen and render rigid throughout by a system of bracing and counter-bracing.

374. For consider what occurs when a train crosses a bridge. The resultant load is continually increasing in amount and altering its position until the whole train has passed on to the bridge or the whole bridge is covered, so that were a flexible chain or arch called on to support it during its passage, its figure would be constantly changing in order to adapt itself to each new position of the load, and always correspond with the ever varying diagram of bending moments.

375. In order, therefore, to prevent such distortion the flexible curve must be rendered permanent by stiffeners, and this is the more urgent theoretically in the case of the arch than in that of the chain, because the state of compressive strain is one of unstable equilibrium.

376. The linear arch, then, being the reverse of the funicular curve, and the bending moments identical under similar conditions of loading, length of span, &c., the distorting tendencies may be counteracted in either case by exactly similar arrangements. They are the following :—

Method I.—The roadway, suspended from the chain, or supported on the arch, may be stiffened by means of an auxiliary girder extending the whole length of the span, as in *Fig.* 135.

Method II.—The arrangement from which the roadway is hung, or

on which it is carried, may itself be stiffened. This may be effected in the following four ways, *viz.*:—

(1). The chain may be braced to the roadway by triangular bracing, as in *Fig.* 134, thus forming a rigid system.

(2). Two flexible chains may be hung, or two arches placed, one below the other and braced together, as in *Figs.* 136 to 138.

(3). The chain or arch may be replaced by two girders or braced ribs, either curved or straight, hinged together at mid-span and each covering half the span, as in *Figs.* 139 to 142.

(4). The span may be crossed by a single girder, pinned at one end and suspended from a link at the other, as in *Fig.* 165. This arrangement is known as the Suspended Girder. The Relieved Bowstring, *Fig.* 166, may also be included in this class.

377. In either case certain distorting stresses, produced by the unequal distribution of the travelling load, have to be provided for, the bending moments being, of course, the same whether resisted by the stiffened chain or braced rib. Now, in the earlier Chapters it has been explained that, a weightless bar being supposed to connect the points in which the resultant line of resistance meets the abutments of a Suspension Bridge or Arch, the bending moment at any vertical section is resisted by a couple which is made up of the stress in this imaginary closing piece multiplied by the distance intercepted between the points in which the section plane in question is cut by the imaginary piece, and the actual resultant line of resistance of the arch or chain. It is obvious, then, that so long as no partial travelling load is on the chain or arch, that is, so long as the platform is uniformly loaded throughout, or practically so, the chain, or linear arch, and abutments are sufficient to resist the bending moments, and therefore, under these circumstances, whatever stiffening arrangement there may be, will not be stressed at all. Immediately, however, a partial load comes on to the bridge, and the loading becomes unsymmetrical, the bending moments at sections in the neighbourhood of the added load become increased, and those at other sections correspondingly diminished. It is to this variation in the bending moments that the tendency to departure of the line of resultant resistance from the parabolic figure is due, and it is to meet these differential bending moments that the stiffening arrangement is required. Hence the conclusion—

The permanent dead load need not be taken into account in calculating the maximum stresses produced in the pieces of the stiffening arrangement; neither need the travelling load, provided it be uniform and cover the whole length of the bridge.

378. If, then, the structure be rendered sufficiently stiff, the parabolic form of the line of resultant resistance will not be sensibly distorted by the passage of the travelling load, and consequently the resultant vertical stress transmitted to it by each vertical connecting rod must necessarily be equal throughout. Such being the case and a partial uniform travelling load having come on to the bridge, there will be an immediate tendency of the vertical rods connected with that portion of the platform which carries the added load to sink, accompanied by a consequent tendency of the remaining vertical rods, (connected with the unloaded portion) to rise. So that it results from what has been said, that, if the parabolic figure be preserved, the nature of the stresses developed in the former rods will be of equal intensity, but of opposite sign to those developed in the latter; and the stiffening arrangement must consequently be counterbraced throughout, each piece being designed to meet upward and downward stresses of equal amount, varying from a certain maximum compression to an equal maximum tension. Moreover, the roadway should evidently be pinned down as well as supported at the abutments to prevent all liability to rise under certain conditions of loading, but on no account should it be fixed, as that would interfere with its free expansibility under variations of temperature.

379. Now although the above remarks are strictly applicable to structures designed according to each of the methods enumerated above, to none of them is its application so evident as it is to the stiffening girder attached to the roadway of the suspension chain, because the other structures being designed to carry both dead and live load together, while the suspension chain girder only comes into action when the live load covers a *portion* of the span, stresses permanently developed in the pieces of the rib by the action of the dead load are scarcely likely to pass through value zero and change their character during the passage of a partial live load, which, as the span increases, bears an ever diminishing ratio to the dead load.

380. Whichever method be adopted, however, care must be taken to

design the principal member supporting the load on one side the roadway, whether it be a suspension chain, pair of braced chains, metal arch, pair of braced or plate ribs, or suspended girder, strong enough to carry its, or their, own weight together with that of half the entire roadway, and of the greatest travelling load which is ever likely to come on to the bridge, added to the weight of the supporting rods and connections on that side of the bridge. And it must be remembered that *the function of the stiffening arrangement is in no way to add to the supporting power of the structure, but simply to distribute the effect of a partial load evenly over the supporting member.*

Hinges.

381. All the structures referred to in para. 376, with exception of the Suspended Girder, are best hinged at mid-span, although the hinge of the Chain Suspension bridge is sometimes placed near one of the abutments, as in *Fig.* 135, which shows a roadway-girder hinged at *t* and connected with the abutment by a short flap which rises and falls with every change of temperature. The roadway is, of course, hinged and not fixed at each abutment.

*Comparison of Methods I. and II.**

382. In comparing the two methods referred to in para. 376 it will be observed that the employment of a flexible chain, combined with a stiffened roadway, as in Method I., is by no means an economical arrangement, because not only must the cross section of the chain be made strong enough to carry the whole load both dead and live, just as if it formed part of a rib, designed as in Method II., but the unequal distribution of a partial load has, in addition, to be provided for by separate girders which would be strong enough of themselves to carry the load more than across half the interval, and the pieces of which are most uneconomically strained, the stresses passing from tension to compression through value zero. It is, therefore, more advantageous in every way to make the supporting member stiff in itself, as in Method II., because by so doing a rigidity is obtained which is greater than that which can be obtained in the roadway girder, and, moreover, as the functions of supporting the load and stiffening the structure are both performed by the same member, material is economized. For if the

* Claxton Fidler's *Practical Treatise on Bridge Construction*, p. 361.

pieces of the structure be designed, as in Method II., to carry the greatest load that is ever likely to come on to the bridge, they will also be able to resist the bending action of a partial load without being dangerously strained.

Suspension Bridges and Braced Arches.

383. Professor Ritter* makes the following remarks in regard to the relative advantages of Metal Suspension Bridges and Braced Arches :—

" Suspension Bridges (for single spans) do not, unless special arrangements are made, compare favourably with Braced Arches, as regards the amount of material employed; for in the latter the points of connection with the abutments are low down and the horizontal thrust acts against the abutments in the direction in which they are strongest; whereas in the former, the points of attachment are placed high up, and the horizontal pull tends to turn the piers over in the direction in which they are weakest; consequently the quantity of material in the piers will be much greater in the one case than in the other.

" The comparison would be less unfavourable to the suspension bridge, if there were several spans, the horizontal tensions neutralizing each other at the central piers, at least when the spans are equally loaded, but the objection still applies to end piers."

Anchorages and Shore Abutments.

384. Illustrations of Anchorages and Shore Abutments are shown in *Figs.* 186 to 188, *Plate* XXXIV., and will be referred to again later on.

Economic Proportions.

385. Mr. Claxton Fidler remarks †: " As in the case of the parabolic girder, the economy of the chain bridge, and its limiting span, may be greatly increased by adopting a more liberal depth." Thus, with a depth of one-eighth the span " the limiting span, with a working stress of 5 tons per square inch, will be increased to 2,560 feet; and if we adopt a depth equal to one-fourth of the span, the limiting span will be theoretically increased to 4,160 feet. But in practice it is impossible to adopt the

* Ritter's *Iron Bridges and Roofs*, Chapter VIII., (translated by Lieut. (now Captain) Sankey, R.E.).
† Practical Treatise on Bridge Construction, p. 258.

most economic proportions or anything like them, because the greater the dip of the chain the greater will be the distortion produced by the rolling load, and the greater the oscillations attending its passage across the bridge. * * * * These reasons have contributed to limit the proportions adopted in practice, and a tolerable degree of steadiness has been attained by stretching the chain to a flat curve. In most of the existing examples the dip does not exceed $\frac{1}{10}$th of the span, and in many cases a depth of $\frac{1}{12}$th to $\frac{1}{18}$th has been adopted with advantage—so far as stability is concerned, but of course with a considerable sacrifice of economy."

Depth of Stiffening Girder.

386 Sir B. Baker fixes the depth of the stiffening girder * at about $\frac{1}{18}$th the span for spans of 300 feet and upwards, to $\frac{1}{27}$th for spans of about 1,400 feet and upwards. For shorter spans than these the proportions given in para. 195, Vol. I., may be adopted.

Method I.

The unhinged Stiffened Chain Suspension Bridge.

387. The stiffened Chain Suspension Bridge is the common example of this method. The roadway girder may be hinged at mid-span, near one abutment, or not at all. The last-named arrangement is objectionable, but, as many existing bridges are constructed on this principle, it will be considered first.

388. As has been already stated, the roadway girder whose function it is simply to stiffen the chain, must not be reckoned on to support the load at its extremities, although necessarily secured there in order to prevent its rising, but be simply regarded as suspended from the chain at certain points by means of the suspenders.

389 In considering the equilibrium of a partial load, therefore, we must regard it as suspended from the chain at the following points, *viz.*, the point of suspension of the chain at the abutment which is the nearer to the centre of gravity of the load, and the suspenders connected with the panel points of the girder.

The value of the reactions at each of these points depends upon the normal form of the chain. If, as is usually the case, the figure be

* Long Span Railway Bridges, p. 55.

parabolic, which is that corresponding to a series of vertical loads equally distributed along a horizontal line, that is, in fact, uniformly distributed loads, the reactions are necessarily equal throughout; if, on the other hand, the normal form be that of any other geometrical figure, then the intercepts on a straight line drawn parallel to the suspenders, generally vertical, across the vectors of the force polygon determine the magnitudes of the corresponding reactions.

We shall, as already explained, suppose the figure of the chain to be parabolic.

The construction for determining the reactions at the points of support due to a partial travelling load, and the stresses developed in the girder will be apparent from the following example:—

*Example I.**

390. A clear interval of 300 feet is to be spanned by a stiffened Chain Suspension Bridge, of the form shown in *Fig.* 169, *Plate* XXXII., the girder not being hinged at the middle of the bridge. The panels of the girder are to be 15 feet apart, and its height 15 feet. It is required to determine—

(1). The stresses in the stiffening girder when an engine and tender only, weighing 1·25 tons per foot run (uniform) come on to rods 12, 13, 14 and 15.

(2) Also the maximum stresses for which the links of the chain are to be designed, taking the weight of the structure at 1·25 tons per foot run, supposed uniformly distributed, and the uniform travelling load covering the whole span at 1 ton per foot run.

Scales.—Lineal Scale, 50 feet to 1 inch

Scale of loads, 20 tons to 1 inch.

(1). *To measure the stresses in the girder.*

In order to measure the stresses produced in the girder by the given partial load, the polygon of bending moments must first be described in the usual way.

Thus the load carried by each of the four loaded suspenders 12, 13, 14 and 15 is $15 \times 1{\cdot}25 = 18{\cdot}75$ tons, making a total applied load of 75 tons

* The general method that follows is taken from Chalmers' *Graphical Determination of Forces in Engineering Structures*, p 293

Set off $A''a''$ downwards from A'' and equal to 75 tons on the scale of loads, and divide it into four equal parts in the points a_1, a_2, a_3. Take any convenient pole P_1 at a convenient distance, say 100 lineal units, from $A''a''$, and describe the polygon of bending moments A'' 12, 13, 14, 15 A' in the usual way. Join A' A'' and divide it into 20 equal parts in the points 1, 2, 3.........19, and through these points draw the vertical lines of loads

If A'' 12 and A' 15 be produced to meet in p_1' and the vertical line p_1p_1' be drawn through it, then would p_1p_1' be the line of resultant partial load, were the chain flexible and weightless.

But the figure of the chain is to remain symmetrical and parabolic under all conditions of the loading, therefore the stiffening girder must take up such stresses as will cause the reactions of the suspending rods to be equal throughout and the resultant load line to be shifted into the vertical plane which divides the bridge symmetrically and contains rod 10.

Therefore, if C_1 is the point in which the side $A''p'$ of the equilibrium polygon cuts the vertical through rod 10, the middle rod of the bridge, then will C_1A'' be the direction of the actual resistance at A'' of the partially loaded chain.

Through P_1 draw P_1a_1'' parallel to C_1A'; then will $a''a_1''$ represent the additional resistance that must be taken at the support A' in order that the chain may not become distorted, and the remaining vertical resistance $a_1''A''$ must consequently be taken up equally by the 19 suspending rods, if the chain is to maintain its parabolic figure.

Divide the distance $A''a_1''$ into 19 equal parts and number them downwards from A''; then will each division represent the resistance of the rod which bears the same number.

By joining the 19 points of division of $A''a_1''$ to the pole P_1 the force polygon of the chain reactions is obtained, and hence the equilibrium polygon of chain reactions can be described in the ordinary way and it will necessarily be parabolic.

Let us examine how the bending moment produced by the partial load is resisted at any section as p_1p_1'. The total ordinate $p_1\pi''$ measures 43·4 tons about on the scale of loads, and therefore represents a bending moment of 4,340 foot-tons, since the pole distance has been taken at 100 lineal units; of this moment, $p_1\pi'$ which measures 17 tons about and therefore represents 1,700 foot-tons, is taken by the chain, the remain-

der $\pi'\pi''$ representing a bending moment of 2,640 foot-tons about is resisted by the stiffening girder. In fact—

The bending moment of the auxiliary girder at any cross section is measured by the number of load units there are in the length intercepted between the cord polygon of chain reactions and that of the partial load, multiplied by the number of lineal units in the polar distance of the force polygon (or vice versâ, i.e., the intercept may be measured in lineal units, and the pole distance in load units).

The several steps, then, are as follows :—

(*a*). Describe the moment polygon of the partial load.

(*b*). Determine the reactions at the abutment which is nearer the line of action of the resultant load, and of the rods.

(*c*). Describe the moment polygon of chain reactions.

The required intercepts can then be measured, and bending moments determined, which, divided by the effective depth of the girder, give at once the stresses in the flanges.

(2). *To measure the maximum Stresses in the chain links.*

In order to measure the maximum stresses to which the chain links are liable, we have the following data —

The load carried by each one of the 19 vertical rods placed 15 feet apart is $15 \times (1{\cdot}25 + 1) = 33{\cdot}75$ tons As one-half this load is taken at each abutment, we have for the total load $33{\cdot}75 \times 20 = 675$ tons.

It is obvious that the force and equilibrium polygons, already drawn for the chain reactions, may be utilized to measure the link stresses, if we simply extend the load line $A''a_1''$ either way by half the length of a single reaction, that is to the points a and β, as shown, and alter the scale of loads. As the total load, represented by length $a\beta$, actually measures 51 tons on the old scale, and should represent 675 tons on the new, we have the following proportion for forming the new scale :—

$$675 : 100 :: 51 = 7{\cdot}55 \text{ tons}.$$

The new distances have been scaled on the underside of the old scale of loads. The pole distance has not been changed and measures 100 lineal units.

The maximum ordinate of the link polygon (formerly the chain reaction polygon) measures 25·5 tons about on this new scale, consequently the bending moment at this cross section amounts approximately to

to 2,550 foot-tons, and the maximum link stress $A''P_1$ measures about 603 tons.

Summary of Construction for design of Unhinged Stiffening Girder.

391. The above example illustrates the method of determining the reactions at the abutments, the chain reactions, and the link stresses. Having drawn a series of equilibrium polygons for different dispositions of the load, it is easy to pick out the maximum moment at each apex or panel point, and these maxima may then be laid off for the respective apices and a diagram of maximum moments obtained for the girder. These being scaled off and divided by the effective depth give the maximum stresses in the flanges.

Bending Moments.—By calculation it may be ascertained that the greatest bending moment occurs when about six-tenths of the girder is loaded with a uniform load. It follows, then, that the maximum bending moment in the stiffening girder is the same as for a simple girder of six-tenths the span loaded with travelling load only.

392. *Shearing Forces.*—The diagram of shearing forces for the partial load dealt with in the previous example is shown by a chain line in *Fig.* 170. Its bounding lines are—from panels 1 to 12 along the line $A''A'$ then down along the zigzag line *abcdefg* to a' and a_1' and so along the stepped line back to panel 1.

It will be observed that the cross section of zero shearing force occurs at panel 13, at which panel the section of maximum bending moment also occurs.

Also, since the active shearing force is the resultant stress on the right side of the section, that on the left being regarded as the shearing resistance, it will be observed that the portion of the diagram to the right of panel 13 represents upward forces and that to the left downward ones, the diagram being, in fact, plotted in the reverse direction to that usually adopted for convenience of representation.

393. It is, moreover, obvious that the maximum shearing force occurs at any chosen cross section of the girder when the uniform travelling load extends from either support up to the section in question, and the longer segment is fully loaded; and further, that it is measured at that section by the sum of the chain reactions in front of the load. The strain in the corresponding diagonal of the stiffening girder is then

known, being equal to the shear at the apex of the panel multiplied by the secant of the angle which the diagonal makes with the vertical.

Hinged Stiffening Girder.

394. When the stiffening arrangement consists of two roadway girders, each covering half the span, and hinged at mid-span to one another, the resistance at the mid-span joint for the unloaded half of the bridge must pass through the point of the abutment pier from which the chain on that side is hung, as shown in *Fig.* 192, *Plate* XXXIV., and the system becomes, in fact, equivalent to two links $a'SA'$ and, $a''SA''$ hinged at the point S, and suspended from the points a' and a''. The case is, in fact, the reverse of the braced arch in many respects, which is dealt with, as far as concerns stresses due to travelling load, in para. 400, *et seq.*

An examination of the effect of different positions of the travelling load, after the manner of the paras. referred to, will show that—

(1). The bending moment at any cross section of the stiffening girder reaches a maximum when the load extends from the abutment beyond the central hinge to a point on the roadway for which the moment at the cross section under consideration is zero—this disposition producing tension in the flange when one of the segments is loaded and compression when the other one is.

(2). And the maximum shearing force, it will be seen, occurs at one extremity of the girder, when the adjacent half span is fully loaded.

The Shore Chains.

395. The shore chains may be designed on the following principles* :—

The directions of the links either side the abutment pier and magnitudes of the link stresses must be such as to produce a resultant acting down the centre of the pier.

The masonry behind the abutment to which the chain is anchored is generally proportioned so as to be capable of resisting at least twice the tension developed in the end link of the bridge chain at the abutment.

If the figure of the shore chain and directions of the stay rods be

* Chalmers' *Graphical Determination of Forces in Engineering Structures*, Chapter VI.

known, then the stresses in the pieces and weight of masonry required can be determined by means of force and cord polygons in the following manner:—

Prolong the direction of the end link C_1A' of the bridge chain, *Fig.* 170, to P_2, making $A'P_2$ equal to twice P_1a_1''. For convenience in *Plate* XXXII., $A'P_2$ has been made equal to P_1a_1'' in length, but is measured on double the scale, so that the weights of masonry blocks must be also measured on the latter or double scale.

Through P_2 draw a vertical straight line, and from A' draw $A'P_3$ parallel to the given direction of the top link of the shore chain, cutting the vertical through P_2 in P_3. Then is P_3 the pole of the required force polygon which can be completed in the usual way by drawing through P_3 vectors parallel to the successive links, and, commencing from A', straight lines successively parallel to the successive stay rods at the points where the latter severally meet the corresponding vectors (not shown in *Fig.* 170, the scale being too small). Then will the weights of the successive masonry blocks be measured by the projections by horizontal rays of the respective stay stresses (in the stay rods) on the vertical through A'.

396. It is evident that the resultant reaction at the abutment pier will be vertical because whatever proportion of $A'P_2$ be called into play at the pier apex the same proportion of $A'P_3$ will be so also, and the resultant P_2P_3 will, therefore, be always vertical.

397. If, on the other hand, it be required to design the shore chains to be weighted in a given manner, then the procedure would be somewhat as follows:—

Suppose there is a length of about 60 feet of stone work, 30 feet wide (giving 15 feet for each set of chains) and $12\frac{1}{2}$ feet deep, and weighing about 151 lbs. per cubic foot behind the abutment. Each stay rod will be weighted with a block $15' \times 15' \times 12\frac{1}{2}'$ weighing about 200 tons.

Extend the direction C_1A' of the last bridge-chain link, as before, backwards, and measure on it a stress $A'P_2$ equal to double that of the bridge-chain link at A'. In *Plate* XXXII. the scale of $A'P_2$ has been taken double that of P_1a_1'', so that length $A'P_2$ is equal to length $a_1''P_1$.

From A' set down the four loads, each 200 tons, and using P_2 as pole, describe the force polygon $P_2\,A'a_2'$, and from $A'a'$ the correspond-

ing cord polygon A′ to e', the apices of which lie on the lines of loads s_1, s_2, s_3 and s_4 of the blocks of stone behind the abutment. Let the extreme sides of this cord polygon meet in G′. Then will the line of action of the resultant of these abutment loads pass through G′.

If now the stay rods are vertical, it is only necessary to project these force and cord polygons (by moving the pole vertically downwards) so that the extreme sides of the latter shall pass through A_1' and s_4, *Fig.* 169. This may be done thus—

Produce the extreme side e'G′ of the cord polygon already drawn to meet the vertical through A′ in D, and set off from A_1' the length $A_1'd$, *Fig.* 169, equal to A′D, *Fig.* 170. Join d with the extreme point s_4 beyond which the chain is anchored, and join $g'A_1'$; then is $g'A_1'$ the direction of the top link at A_1' of the shore chain, and ds_4 (or $g's_4$) is that of the extreme side of the required cord polygon, *i.e.*, of side 5.

From a_2', *Fig.* 170, draw a_2' P_3 parallel to ds_4, *i.e.*, side 5, and from A′ draw A′P_3 parallel to $A_1'g'$; then will P_3 be the new pole, and it will be the same distance from the load line as is P_2. The force and cord polygons can now be drawn in.

If the stay rods are inclined to the vertical, as is generally the case, then a portion of the horizontal pull will be taken by them, and the force and cord polygons will be correspondingly altered, but the directions of the extreme sides and the line of action of the resultant loads will not be altered.

Fig. 187 shows a hollow anchorage in the form of an arch, suitable for anchoring side 5.

Method II.

398. As remarked in para. 381, all the structures designed under Method II. are hinged at each abutment, and also at mid-span or near one of the abutments, so that each structure constitutes a linked system.

In the Hinged Stiffened Suspension and Arched Rib the length of each link or rib is the same, the hinge being at mid-span, and each rib being in fact a suspended girder, consisting of an upper and lower member united by a plate web or by open bracing, and each deep enough to ensure the requisite degree of rigidity. About one-half the rise of the arch, or depth of the central hinge of a suspension bridge below the pier hinge level, would usually be a sufficient maximum depth for the rib. This

system may be designed in many forms, but it will be sufficient to call attention to the following, *viz.* :—

(1). The form shown in *Figs.* 137 and 138 in which each rib is a parallel girder, whose neutral axis corresponds with the parabolic form of the suspension curve or lineal arch.

(2). The form shown in *Figs.* 141 and 142 in which the depth of each rib is proportional to its curve of bending moments, each being, in fact, a parabolic girder.

The latter form is by far the better of the two.

(3). The form shown in *Figs.* 139 and 140, consisting of braced ribs, hinged to one another and also to the abutments.

(4). In the suspended girder, *Fig.* 165, the length of one of the links or ribs of the system reaches a minimum.

399. If the bridges of this system be designed to carry a full load, they will be strong enough to carry a partial one, and as the stresses in none of the pieces will fall short of that due to the dead load alone, they will not alter in their nature during the passage of the travelling load, as in the case of the stiffening girder to the suspension cable, and are consequently strained to much greater advantage.

400. In the case of bridges covering long spans, however, economy of material is of the utmost importance, and the following method * of treating the Braced Arch and Suspension Rib illustrates the manner in which that distribution of the travelling load may be ascertained which produces the maximum stress in each piece of the structure.

401. *Fig.* 171, *Plate* XXXIII., shows a Braced Arch with curved bottom boom, hinged at mid-span and at each abutment, and containing 10 panels in each semi-span.

Let the several pieces of each panel be lettered as shown. It will be sufficient to examine the distribution of load as affecting the left half span only.

Since the system is equivalent to the two links A″S and A′S, which are freely hinged at their extremities, a load placed any where on the left half of the arch will produce a hinge reaction at S which is entirely in direction A′S; and similarly one on the right half in direction A″S.

With this preliminary remark we shall proceed to examine the distri-

* *Vide* Ritter's *Iron Bridges and Roofs.*

bution of loads producing a maximum stress in each piece of the left semi-arch.

402. *Stresses in Bars* X.—Consider bar X_5, *Figs.* 171 and 172, *Plate* XXXIII. It is necessary to determine which loads produce tension and which compression, and to do this, since the loading is, by hypothesis, continuous, the cross section must be determined at which the load if placed produces no stress at all in X_5. Because, regarding the continuous travelling load as a series of very small equal detached loads closely following one another and passing continuously from right to left (suppose) across the bridge, it is evident that once the head of the load has passed to the left side of this cross section, the value of the bending moment of that portion of the load which is to the left of the section has already changed sign, that is, the resultant bending moment at the section in question has begun to diminish in numerical value.

Let any vertical plane $\alpha\beta$ then be taken cutting the three bars X_5, Y_5 and Z_5. Then, the point about which to take moments in order to determine the stress in X_5 is clearly the point of intersection of the remaining two bars, namely L. (*Figs.* 171 and 172).

If A″L be joined, *Fig.* 172, and produced to meet the direction of the hinge reaction R′ (acting in direction A′S) in C, then will the vertical plane containing C be that in which the weight Q must hang so as to produce zero bending moment at cross section $\alpha\beta$.

For, if Q be hung in that vertical plane, it must be balanced by reactions in directions in A′C and A″C.

Thus, the portion $\alpha\beta$S of the arch is acted on by the impressed forces Q and R′, which are equivalent to the single resultant R″ acting in direction CA″, which, therefore, has no moment about L, being a point in its direction. Therefore also is no resistance called into play in X_5, there being no bending moment requiring to be balanced. If Q be placed to right of C, then its effect for all positions to right of S is along hinge reaction R″ in direction SA″, and for all positions between S and C along directions of R″ lying between SA″ and CA″ and passing below L. So that the effect on $\alpha\beta$S is to produce a moment acting in the direction of the hands of a watch, and therefore extending X_5.

For positions of Q between C and $\alpha\beta$, R″ passes above L, and there-

fore produces a moment round L in a direction contrary to that of the watch hands, and therefore compressing X_5.

For positions of Q to left of $\alpha\beta$, the resultant of Q and R″ is the force R′, acting on $\alpha\beta$S in direction from S to A′, and producing a moment round L tending to compress X_5.

The nature of the stress produced in X_5, corresponding to the position of the load in regard to section $\alpha\beta$, is indicated in *Fig.* 173.

It has been already explained that for an arch or suspension bridge the curve of whose system of pieces immediately sustaining the loads is an equilibrated one, as soon as the bridge is fully loaded, the stresses in the horizontals and diagonals disappear, the verticals transmitting directly to the pieces of the bridge immediately sustaining the loads. Now, as explained, the permanent load is uniformly distributed over the span and, therefore, produces no stress in the horizontals or diagonals.

Thus, in calculating the stresses in horizontals and diagonals the permanent load may be left out of account.

403. *Stresses in Bars Z.*—Consider bar Z_5, *Fig.* 172, and, as before, let a vertical plane $\alpha\beta$ cut the three bars X_5, Y_5 and Z_5. Join A″ to N, the upper extremity of the diagonal Y_5, and produce to meet R′ in B. Then, from what has been already said, it is easy to see that the vertical plane containing B is that in which the load Q must be hung in order to produce zero bending moment in Z_5 at any cross section $\alpha\beta$, the point N being clearly that about which to take moments; and, further, that positions of Q to the right of B cause compression, and those to the left, tension in bar Z_5. There will, in fact, be a maximum tensile stress in Z_5 when the segment of the platform from abutment A″ up to B is fully loaded, the right segment from B to abutment A′ being unloaded, and a maximum compressive stress when the right segment is loaded and the left unloaded.

And it must be remembered, as before explained, that the permanent load must be taken into account in designing the pieces Z. The nature of the stress produced in Z_5 by the load in various positions is shown in *Fig.* 173.

404. *Stresses in diagonals* Y.—Consider bar Y_1. Taking the vertical section $\alpha\beta$ as before, the point about which to take moments is clearly M_1, the point of intersection of the other two sides (Z_1 and X_1) cut by the plane. (*Figs.* 171 and 172, *Plate* XXXIII).

Join $A''M_1$ and produce it to intersect R' in D_1; then Q must obviously hang in the vertical plane containing D_1 in order to produce zero moment about M_1.

For, as before explained, Q and R' balance about M_1, being a point in the direction of their resultant; and consequently the moment of the stress in Y_1 must be zero, that is, Y_1 will not be strained at all.

If Q be placed to the right of S, the resultant of Q and R' will pass in direction SA'', and if hung in any vertical plane between S and D_1 in some direction lying between SA'' and D_1A'', in either case passing below M_1 and producing a moment round M_1 acting in the same direction as the hands of a watch, and therefore compressing Y_1.

If Q be hung in a vertical plane between D_1 and $\alpha\beta$, the resultant R'' passes above M_1 and produces a moment in a direction contrary to the hands of a watch, and therefore tending to extend Y_1.

If Q be hung to the left of section $\alpha\beta$, the resultant R', acting in direction SA', produces a moment round M likewise tending to extend Y_1.

These results are shown in *Fig.* 174.

In the case of the diagonals of panels which are so situated that M falls to the right of e'' (or left of e', as the case may be), *Figs.* 171 and 172, the point M lies to the right of, that is above, R' (or left of R'' as the case may be), and there will then be three groupings; that is to say, as the detached load moves on to the bridge from right to left (suppose), the stress in these diagonals will twice pass through value zero.

Consider the diagonal Y_5, *Fig.* 172. The position D_2 (being the intersection of $A''M_2$ with R') of the section of zero moment at $\alpha\beta$ can be found as before, the resulting stresses in Y_5 for loads hung on the span from A' to D_2, and again from D_2 to section $\alpha\beta$, being the same as in the last case, *viz.*, compression in the former case and tension in the latter. At section $\alpha\beta$, however, there is a change in the nature of the stress, for immediately the isolated load passes to the left of the vertical plane $\alpha\beta$, the active resultant force influencing the portion $\alpha\beta S$ changes to R' which now passes below, instead of, as before, above M, the result being a compressive instead of a tensile strain in diagonal Y_5.

These results are shown in *Fig.* 174. Thus, a maximum compressive strain is developed when the bridge is loaded from A' to D_2, the strain being again compressive when the segment between A'' and section $\alpha\beta$ is fully loaded, the remainder of the bridge in each case being unloaded.

A maximum tension in the diagonals under consideration is obtained when the segment extending from section $\alpha\beta$ to the vertical plane through D_2 is fully loaded, the remainder of the span being unloaded.

For diagonals, such as Y_9 and Y_{10} for which M falls at a more or less infinite distance, the section itself, such as $\alpha\beta$, is the only limit (*Fig.* 171).

As already explained, the permanent load need not be taken into account in determining the stresses in the diagonals.

405. *Stresses in verticals* V.—In examining the verticals, the section plane $\alpha_1\beta_1$ must be oblique, not vertical, otherwise the vertical would not be cut through (*Fig.* 172).

The load limits for V_6 are the same as for Y_5, and in the case of panels near the central hinge there is, as for the diagonals, a second limit, *viz.*, the oblique section $\alpha_1\beta_1$ cutting the vertical, so that the position of the intersection of this oblique plane with the roadway will differ from that of the limiting plane of the diagonal by one bay. The conditions of loading that produce tension in the diagonal will be found to produce compression in the corresponding vertical, and *vice versâ* (*Fig.* 174).

The effect of the permanent load on the verticals must, as already explained, be added algebraically to that of the moving load.

The Shore Spans of Suspension Bridges.

406. Professor Ritter, in Chapter VIII. of his *Iron Bridges and Roofs*, points out that the objections made by him to the employment of Suspension Bridges for single spans as compared with Braced Arches, and quoted in para. 383, are obviated by the adoption of short shore spans, as shown in *Fig.* 175, and he proceeds to investigate the effect of the travelling load on a structure of this form in the following manner:—

Such a bridge can, on the whole, be represented by the system of links shown in *Fig.* 176, a vertical load placed between A and C or B and D producing vertical reactions at A or B; one placed between A and B, reactions either in directions AC or BD, according as the load is between A and S, or S and B, S being the position of the central hinge of the main span AB.

Thus, a load placed on either of the end links AC or BD has *no effect* on either of the remaining links.

"When the rod CA, therefore, is alone loaded, it behaves like an ordinary beam supported at both ends, and when the rods AS and SB are loaded, they are in the same condition as if the points of suspension A and B were fixed." *

As the system of four braced portions is hinged at the points C, S, and D, and freely suspended and hinged at A and B, it follows that the stresses in the bars of the centre, or main, span can be determined in the manner already explained for hinged arch work, while those in the bars of the shore spans will now be investigated, the two following laws being observed, *viz.* :—

(1). A load on the central span necessitates a reaction at the points A and C in direction AC, and similarly for span BD.

(2). A load placed on one of the shore spans produces a vertical reaction at A or B as the case may be. The shore spans may each be equal to one-half the main span length.

407. *Stresses in horizontals* X *of Shore Spans.*—The stress in any horizontal X_5 is to be found by taking a vertical section $\alpha\beta$ cutting the three bars X, Y and Z, and then forming the equations of moment either for part $\alpha\beta$C or $\alpha\beta$A, with reference to J, the point of intersection of the other two sides, *vide Fig.* 177, *Plate* XXXIII.

Any vertical load placed on any point of the roadway of C$\alpha\beta$ produces, by law (2), a vertical reaction at A and necessitates a tensile stress in X_6.

Any vertical load placed on the portion $\alpha\beta$A produces similarly a vertical reaction at C, and consequently a tensile stress likewise in X_6.

A vertical load placed on the roadway of the central span produces a reaction at C in direction AC, and consequently a compressive stress in X_6.

A load on half span BD has no effect on half span CA. Therefore the groups are as shown in *Fig.* 178.

When the load is uniformly distributed throughout, as before explained, the horizontals and diagonals are under no stress if the curve of the chain be parabolic.

408. *Stresses in the diagonals.*—In order to measure the stress produced in a diagonal, moments must be taken about the point of intersection of the other two sides cut by the chosen vertical plane. Consider

* Ritter's *Iron Bridges and Roofs*, p. 166.

the panel whose diagonal is Y_6, *Fig.* 177. Produce bars X_6 and Z_6 to meet in M, as before. A load placed on $A\alpha\beta$ produces a vertical reaction at C and consequent compression in Y_6; one on $C\alpha\beta$ likewise compression in Y_6; one placed on the central span, a reaction in direction AC and consequent tension in Y_6. The general effect, therefore, is the reverse of that produced in X_6, (*Fig.* 178).

For diagonals of panels situated so near the abutment that the point M falls between the vertical through abutment hinge C (or D) and the line of reaction AC (or BD), the moment of the hinge reaction at the abutment obviously becomes reversed, and we have compression where we had tension before in the diagonal for loads placed on the segment between hinge C (or D) and the section in question. Such is the case with diagonal Y_7, the varying stresses in which are shown in *Fig.* 180.

In the case of panels still nearer the abutment, when M falls outside both reaction AC (or BD) and also the vertical through C (or D), the stresses in the diagonal, such as Y_8, remain the same as in the previous case, Y_7, for loads placed on span AC, but all loads coming on to the central span cause compression instead of tension in Y_8, as shown in *Fig.* 179, the moment of the reaction AC being obviously the reverse of what it was in the cases Y_6 and Y_7.

Lastly, for panels so near the abutment that the sides X and Z are so nearly parallel that M falls at an almost infinite distance from the section in question, all loads on span AC produce tension.

409. *Stresses in verticals* V.—The points about which moments are taken in order to determine the stresses in the verticals are the same as those for the diagonals, and the loading boundaries will, therefore, be nearly the same, (the section $\alpha\beta$ being in this case oblique); but it will be found that the loads that produce tension in the diagonals will cause compression in the verticals, and *vice versâ.* Moreover, it will be found that the third and fourth classes of panels, above referred to, (V_7 and V_8) possess a second boundary, determined by the section plane itself, the intersection of which with the roadway will probably not coincide with that of the loading boundary for the corresponding diagonal, owing to the obliquity of the section.

410. *Stresses in the chain links*, Z.—Taking moments about O, the point of intersection of the two sides X and Y, it will be seen that loads on the central span extend bars Z, while those on a side span compress

them. *Fig.* 182A shows the state of stress in bar Z_8 of the span AC, for different positions of the load.

Attachments of Chain to Central Piers.

411. *Fig.* 183 shows an unbraced freely-jointed frame, resting on the pier at the road level; in *Fig.* 184 the top of the pier forms the roller path for two friction rollers, over which the chain is carried; while in *Fig.* 185 the chains are shown as being attached to a smooth metal plate which is carried on friction rollers at the top of the pier.

The above are taken from Ritter's *Iron Bridges and Roofs*, p. 182.

Stability of the Shore Abutments.

412. It has been explained in para. 406 that loads on the central span of the Suspension Bridge shown in *Fig.* 175 produce a pull at the abutment in direction CA or DB, as the case may be, while those placed on the side spans cause vertical reactions at A and C, or B and D.

The former loads, then, tend to overturn the abutment about its lower edge, while the latter add to the stability of the abutment, since they tend to prevent sliding. The stability of the abutment will thus be in the least favourable condition when the central span is fully loaded, and the side spans unloaded.

The suspension chain should be securely attached to the abutment pier by means of a chain built into the masonry and anchored down in it. (Ritter's *Iron Bridges and Roofs*, p. 187).

Stability of Abutments and Piers of Braced Arch.

413. Herr Ritter points out that the force tending to overturn the abutments or piers of an arched bridge is the horizontal component of the thrust of the arch, the vertical component adding to its stability, and that although both these components are greatest when the bridge is fully loaded, yet that the excess of the moment of the horizontal component about the outer edge of the abutment over that of the vertical may reach its maximum with a partial load.

414. In order to determine the position which a load must occupy on the bridge in order to produce no overturning effect, join F, *Fig.* 186, the limiting point at the foot of the abutment beyond which the compressile action of the resultant thrust would be unsafe, with A′, the abutment bridge hinge, and produce FA′ to meet A″S, the direction of

the central hinge reaction of the other half of the arch, in E. A load hung in the vertical plane containing E is in the limiting position; loads hung to the left of E producing resultant thrusts falling outside, that is, to the right, of F, and those placed to the right, producing resultants falling to the left of F.

The most disadvantageous arrangement of loading, therefore, is obtained when the bridge is loaded uniformly from the left support up to the vertical plane containing E.

To resist the effect of this, there is the moment of the weight of the abutment or pier, and in the latter case, that also of the thrust of the contiguous arch (which, however, should be regarded as carrying no travelling load at all).

The Relieved Bowstring and Suspended Girders.

415. An attempt has been made to combine the advantages of the Suspension Bridge and the Girder in the arrangement known as the "Suspended Girder," and also in a series of "Relieved Bowstring Girders."

416. From what has been said in the earlier Chapters of Part I., it is obvious that an upright bowstring girder might be converted into an arch by severing the lower or tension member, and allowing the ends to rest against abutments capable of resisting their horizontal thrust; further, that if the abutments can only resist a portion of the horizontal thrust, and the lower cord be retained, the latter will be subjected to a stress which is measured by the difference between what it would have been had the girder been freely supported at its extremities, and the relief afforded by the abutments; also that in a similar manner an inverted bowstring girder might be converted into a more or less complete suspension structure.

417.* Let, then, the point C of an inverted parabolic bowstring girder, *Fig.* 165, *Plate* XXXI., be suspended from S by the short link CS, which is inclined in the direction of the tangent to the curve of the bow at C and connected with a backstay which is anchored back in heavy masonry, the other extremity A of the bow being either hinged to the abutment in the usual way, or anchored back similarly to the link CS. The parabolic bow and backstay thus form together a suspension bridge

* Claxton Fidler's *Practical Treatise on Bridge Construction.*

which obviously carries a full uniform load without any assistance from the upper boom, because, when the uniform load is distributed over the whole of the girder, the horizontal component of the pull in the inclined link CS will be exactly equal to the thrust which would be developed in the upper member of the inverted bowstring were it freely supported at each end, while the diagonals, too, remain unstrained.

With a partial load, however, the curve of equilibrium necessarily alters, and therefore the direction of the tangent at C, whereas that of the link CS remains of necessity constant. To meet these strains due to partial loads a certain amount of material must be put into the compression bar and diagonals of the bowstring, but it will be observed that the stresses in the parabolic member, as do those in the web bracing and vertical posts, remain unaffected by the pull of the inclined link, and are the same as would be developed in a similar bowstring hinged at its extremities. By this arrangement, however, a large mass of the material that would have been otherwise required for the compression member is dispensed with, while the rigidity and freedom of movement under changes of temperature characteristic of the bowstring girder is retained. The stresses, in fact, which are developed in the straight boom of the girder, the only piece directly affected by this arrangement, under partial loading, are measured by subtracting the horizontal component of the stress developed in the inclined link from the stress that would have been developed in this boom had the girder been freely suspended at its extremities. This arrangement is known as the *Suspended Girder*.

418. The same principle is applied to a series of upright bowstrings covering consecutive spans, *Fig.* 166, the thrust of each bow being met by the counter thrust of the neighbouring one, just as in a series of masonry arches. Under uniform loading throughout the bridge, the lower members will obviously be relieved of all tensile stress.

To produce this result it is necessary so to arrange that the load shall in some way call forth the necessary reaction of the abutments, and this may be done by severing the lower cord at any point in a span. For instance, the central span might be arranged as a hinged rib, and if there were three similar girders covering equal spans, as in *Fig.* 166, the horizontal tie would suffer no stress at any point so long as the load were uniform over the whole bridge. The permanent load, therefore, would produce no stress in the horizontal ties, but a detached load

placed on the central span would produce compression in the horizontal tie of a side span, and if placed on that side span, a tensile strain.

419. A series of inverted bowstrings coupled together over the piers, so that the pull of one bow is transferred to the bow of the next span, might be similarly arranged as a suspension bridge.

Cantilever Bridges.

420. In para. 353 it is explained that by introducing hinges near the points of inflexion of continuous girders, the positions of the latter are mechanically fixed, by which arrangement, while the efficiency is in no way impaired, the calculations in regard to the stability of the bridge are very much simplified. The continuity of the girder might, for instance, be broken at midspan by means of a hinge, when the system would consist of double cantilevers, each supported on a pier at its centre and hinged at its extremities, as in *Figs.* 155, 158 and 161, *Plate* XXXI. If a hinge be located a short distance to right and left of each pier, thus giving two hinges in each span, the system becomes a girder resting on cantilevers, the portions of the girders over the piers acting as supports or cantilevers to the other parts, which are, in fact, girders resting on and hinged to the extremities of them, as illustrated in para. 355. On this latter system a very large number of modern long span bridges are constructed, including the Forth Bridge and that over the Indus at Sukkur. *See Plate* XXXI., *Figs.* 159, 162, and 164.

421. It will be interesting to examine what sub-division of the whole span will give the least quantity of material in the bridge, and the following method of treating the question will be found in *Ritter's Iron Bridges and Roofs.*

He observes that by comparing the stresses obtained for any given case, it will be found that those developed in the diagonals and verticals are small compared to those of the booms; also that the stresses in the curved part of the boom do not differ materially from one another nor from the stress developed in the horizontal boom. Now, as the quantity of material can be taken as nearly proportional to the stress, it follows that by far the largest quantity is contained in the booms, and that it may be regarded, as far as the question under consideration goes, as practically equally distributed between them. Hence, it cannot be very far from the truth to assert that the quantity of material in the

bridge is proportional to that contained in the horizontal boom. The problem therefore resolves itself into the following one:—Required that sub-division of the span which gives the least quantity of material in the horizontal boom. We shall first determine the most advantageous positions of the hinges in the central portion, and then their most advantageous position in the side spans, for permanent load only.

422. Turning to *Fig.* 191, *Plate* XXXIV., consider the equilibrium of the portions A_1AC and CSE. Put H_1 in the former, and H in the latter, for the resultant horizontal stress.

Let the side span measure *a* feet, the central portion being $2z$ feet long, and let the central spans of the bridge measure $2l$ feet, and their central portions, between the hinges, $2x$ feet.

Then, putting w for the intensity of the permanent load and taking moments about S for the portion CSE, and about A_1 for the portion AA_1C, of any central span, we have

$$H \times h = wx \times x - wx\frac{x}{2} \quad \text{.....................(1)},$$

$$H_1 \times h = wx\,(l - x) + w\,(l - x)\left(\frac{l - x}{2}\right) \quad \text{............(2)}.$$

Now, the sectional area of the booms can be found by dividing the stress in them by s, the safe intensity of stress; hence, if A and A_1 be the respective areas we have

$$A = \frac{H}{s} = \frac{wx^2}{2hs} \text{ and } A_1 = \frac{H_1}{s} = \frac{w\,(l^2 - x^2)}{2hs} \quad \text{......(3) \& (4)},$$

and, multiplying each area by the length of the corresponding boom, we obtain the quantity of material in it, thus

$$Q = Ax = \frac{wx^3}{2hs}, \quad \text{.................................... (5)}$$

$$Q_1 = A_1\,(l - x) = \frac{w\,(l^2 - x^2)\,(l - x)}{2hs}, \quad \text{........... (6)}$$

and the total amount in the two booms, making up half the central span, is, therefore

$$Q' = Q + Q_1 = \frac{w}{2hs}\,(l^3 - l^2x - lx^2 + 2x^3),$$

in which x is the only variable. Differentiating, with respect to x, we have

$$\frac{dQ'}{dx} \propto (-\,l^2 - 2\,lx + 6x^2)$$

and equating to zero, we have $\frac{x}{l} = \frac{1 + \sqrt{7}}{6} = 0{\cdot}6076$ about, $= \frac{CE}{A_1E}$

and $\frac{l-x}{l} = 0{\cdot}3924$; hence we obtain the required relation

$$CE : CA_1 :: x : (l - x) :: 0{\cdot}6 : 0{\cdot}4 :: 3 : 2$$

and $$Q' = 0{\cdot}47184 \frac{wl^3}{2sh}.$$

423. For the side spans, the quantity of material (U) in the horizontal boom DF of the semi-central portion of the side span can be found by substituting z for x in equation (5) above, and that (U_1) in the boom DB by writing l_1 for l and z for x in equation (6).

Hence $$U = \frac{wz^3}{2hs} \text{ and } U_1 = \frac{w\,(l_1^2 - z^2)\,(l_1 - z)}{2\,hs},$$

or, writing $(a - z)$ for l_1, where a is the total length of the side span, we have $$U_1 = \frac{wa\,(a - 2z)^2}{2\,hs}.$$

The quantity of material in the length B_1G, that is in $(2DF + DB_1)$ is, therefore, equal to

$$2\,U + U_1 = \frac{w}{2\,hs}\,(a^3 - 4a^2z + 4az^2 + 2z^3) = Q'' \text{ say,}$$

whence, differentiating and equating to zero, we have

$$\frac{z}{a} = \frac{2}{3}\,(-1 + \sqrt{2{\cdot}5}) = 0{\cdot}3875 \text{ about.}$$

Hence $$\frac{a - 2z}{2z} = \frac{B_1D}{DG} = \frac{0{\cdot}22}{0{\cdot}78} = \frac{1}{4} \text{ about,}$$

and $$Q'' = 2\,U + U_1 = 0{\cdot}16706\,\frac{wa^3}{2\,hs}.$$

424. *Proportion of central to side spans.*—We have for the total quantity of material required for half the total span L

$$M = Q' + Q'' = \frac{w}{2\,hs}\,(0{\cdot}47184\; l^3 + 0{\cdot}16706\; a^3)$$

writing $(L - a)$ for l we have the relation in terms of the single variable a, thus, for a minimum

$$M = \frac{w}{2\,hs}\left\{0{\cdot}47184\,(L - a)^3 + 0{\cdot}16706\; a^3\right\}$$

$$\therefore \frac{dM}{da} = -\,3 \times 0{\cdot}47184\,(L - a)^2 + 3 \times 0{\cdot}16706\; a^2 = 0,$$

$$\therefore \frac{a}{L - a} = \sqrt{\frac{0{\cdot}47184}{0{\cdot}16706}} = 1{\cdot}6806,$$

or, putting b for the breadth of the pier AB, and $(l + b)$ for $(L - a)$ we have the relation

$$\frac{a}{2\,(l + b)} = 0{\cdot}8403.$$

425. *Breadth of Piers.*—The portion of the girder resting on a pier must be regarded as supported at two points, and in order that there may be no chance of overturning with a partial load, the distance apart of these two points, or breadth of the pier, must not fall short of a certain length which may be found as follows:—

The worst distribution of the load is that shown in *Fig.* 190, *Plate* XXXIV. Putting w for the permanent and w' for the moving load intensity, and taking moments about B, *Fig.* 190, we have, with the symbols as figured

$$(w + w')\,xz + (w + w')\,z \times \frac{z}{2} - wx\,(z + b) - w\,(z + b)\left(\frac{z+b}{2}\right) = 0$$

solving, and putting $n = \frac{w'}{w}$ we have for breadth of pier b,

$$b \geqq -(x + z) + \sqrt{(x + z)^2 + 2nz\left(x + \frac{z}{2}\right)}.$$

Now, since the ratio n generally increases inversely as the span, it follows that for very small spans, very broad piers would be required. If the part of the girder lying on the pier be anchored down with tension bars, be tied down in fact, at each point A and B, we have equilibrium maintained by means of a pull in direction and amount K, *Fig.* 190, under which circumstances the condition of equilibrium becomes—

$$(w + w')\,xz + (w + w')\,\frac{z^2}{2} - wx\,(z + b) - p\left(\frac{z + b}{2}\right)^2 = \mathrm{K}b.$$

If this arrangement be adopted, and Q be the weight of the pier, we must have for its stability, taking moments about F, *Fig.* 190,

$$\mathrm{Q} \times \frac{b}{2} + wx\,(z + b) + w\,\frac{(z + b)^2}{2} \geqq (w + w')\,z\left(x + \frac{z}{2}\right).$$

If, however, Q becomes greater than is thought advisable, b can be increased by the employment of double piers.

426. Herr Ritter points out that the above calculations are only *approximate*, and that the results, therefore, may be slightly altered to accord with convenience of construction, the spans being made symmetrical as far as possible. Also that the horizontal stresses above arrived at are the same as the maximum stresses in the booms of parallel girders, and that although the stresses in these booms *decrease* outwards towards the abutments, yet that this is compensated by the *increase* in the diagonals and verticals outwards from the centre, so that on the whole the quantity of material in parabolic and in parallel boomed

girders may be assumed to be the same approximately for the same length of span.

Domes of iron Frame-work.

427. The principles on which the stability of hemispherical and spheroidal domes of iron or steel frame-work is examined are similar to those described for arch-work, the dome being supposed cut by a series of vertical planes containing the axis of the dome or spheroid and dividing it up equally between the ribs, and the segment to be dealt with being considered with reference to the vertical plane dividing its rib symmetrically.

It will be perceived that the magnitudes of the loads borne at the several joints of the rib increase, at first rapidly, from crown to springing, each being, in fact, the weight of a certain portion of annulus of roof covering.

Pin and Link Joints.

428. The strength of this link joint is examined in para. 363, Vol. I. Formerly the eye was made by bending the end of the bar round, and welding the end to the main part; now, however, weldless links are usually employed for important work, the end of the bar being flattened and bored.

429. The maximum dimensions in English practice of wrought-iron links are those of Mr. Berkley, *Fig.* 225, *viz.*—

Width of shank,	$B = 1\cdot00$ (taken as unit).
Diameter of pin,	$D = 0\cdot75$.
Width of metal across eye,	$b + b = 1\cdot25$.
" " behind eye,	$E = 1\cdot00$.
Radius of shoulder,	$r = 1\cdot00$.
" " neck,	$R = 1\cdot50$.

430. According to American practice, Mr. Shaler Smith has given the following rules for the ratios of the parts of wrought-iron tie-bars :—

The proportions will depend partly upon the mode of manufacture, and these proportions will again be modified whenever it may be necessary to use a pin whose diameter is greater than about 0·75 the width of the bar. When the size of the pin is a matter of unconstrained

choice, its diameter should be between 0·66 and 0·75 times the width or diameter of the bar, if the greatest efficiency is to be sought.* The best proportions for "hammered" and for "weldless" eyes are shown in *Figs.* 227 and 229. *Figs.* 226 and 228 show the altered proportions when the bar is relatively narrow. In "hammered" eyes the width E behind the pin = B, but the ratio $\frac{b}{B}$ depends on the diameter of the pin. As compared with Mr. Berkley's rules, the proportions (for ratios D = 0·75 B) give more metal at the side of the eye, and if we compare *Fig.* 228 with *Fig.* 227, it would appear that the metal is better distributed in the former than in the latter. Each side of the head is formed by a circular arc described about a centre situated in the pin's axis, and if the radius of this arc is represented by r, then that of the neck R = $1\frac{1}{2}r$. The heads of the "weldless" eye bars, *Figs.* 226 and 227, are circular and concentric with the pin, so that E = b, and the radius of the neck R = $1\frac{1}{2}r$.

431. The following Table is formed from that given by Mr. Shaler Smith, and may be used in designing pin and eye joints :—

Table of Dimensions of Eyes of Iron Tension Bars.†

Ratio of diameter of pin to width of bar (not $\not<$ 0·67) $\frac{D}{B}$	Ratio of width of metal at side of eye to width of bar $\left(\frac{b}{B}\right)$.		Ratio of maximum thickness of bar to width (pin in single shear).	Remarks.
	Hammered eye (section at back of eye = that of bar).	Weldless eye (section at back of eye = that at sides).		
0·67	0·66	0·74	0·21	These ratios apply to flat bars, with eyes of same thickness as bars. If the bars are round, or not of same thickness as eyes, the ratios of *widths* in columns 2, 3 and 4 will represent ratios of sectional areas.
0·75	0·67	0·75	0·25	
1·00	0·75	0·75	0·33	
1·25	0·76	0·80	0·54	
1·33	0·79	0·85	0·61	
1·50	0·83	0·93	0·70	
1·75	0·84	1·00	0·88	
2·00	0·88	1·13	1·08	

432. The following examples will explain the method of application of above Table† :—

* Claxton Fidler's *Practical Treatise on Bridge Construction* p. 223.

† Cols. Wray and Seddon's *Instruction in Construction*, 3rd Edition, p. 168.

If a 4″ bar is attached to a 3″ pin, the ratio of diameter of pin to width of bar is 0·75; therefore width of metal at side of weldless eye would be (from Table) 4″ × 0·75 = 3″; and maximum thickness of bar 4″ × 0·25 = 1″ if pin be in single, or 2″ if in double, shear.

Were the pin 7″ in diameter, the ratio of diameter of pin to width of bar would be 1·75, which would necessitate a width of metal at side of eye of 4″ × 1″ = 4″, and a maximum thickness of bar of 4″ × 0·88 = 3·52″ if in single, or 7″ if in double, shear.

433. Mr. Claxton Fidler remarks* "when wrought-iron eye-bars are formed to the proportions given above, it has been proved by repeated experiments that the full theoretic strength of the shank will be developed; and when the bar is strained by a load equal to its theoretic breaking weight, it will be as likely to give at the shank, as at any other point. At this point the *average* intensity of stress in the section b will, of course, be less than that in the shank in the proportion $\frac{B}{b+b}$; but the *maximum* fibre stress at the inner edge of the section b will probably be equal to the stress in B or nearly so."

"It does not by any means follow that the proportions determined by experiment for wrought-iron eyes will apply also to steel, and it has been remarked that some difficulty has been experienced in the formation of steel eyes, which seems to indicate that a different set of proportions may yet have to be found to suit this material."

434. When many links are carried on the same pin, as in a large bridge, it is obvious that the pin may be regarded as a beam transversely loaded and liable to bend and shear. *Figs.* 230 and 232 show the diagram of bending moments for two different arrangements of the same loads, and illustrate how much the dimensions of the pin are dependent on the arrangement of the links. It is, in fact, advisable to arrange the links so that the stresses shall come alternately in different directions on the pin; (the corresponding force diagrams are shown in *Figs.* 231 and 233).

The shearing stresses may, for all practical purposes, be regarded as uniformly distributed over the cross section of the pin.

Screw Joints and Connections.

435. Screw bolts are employed to resist stress in the direction of the

* *Practical Treatise on Bridge Construction*, pp. 225 and 226.

Fig. 171.

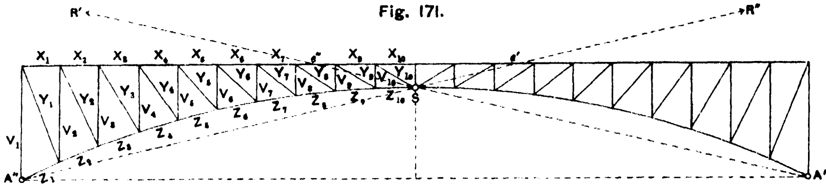


Fig. 172.

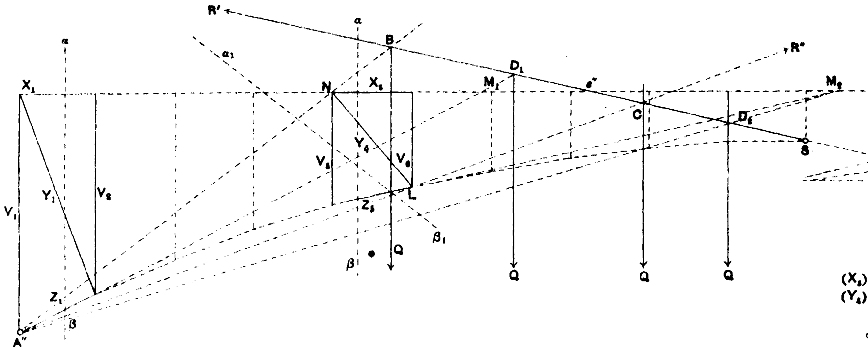


Fig. 173.

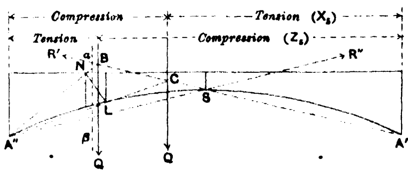


Fig. 174.

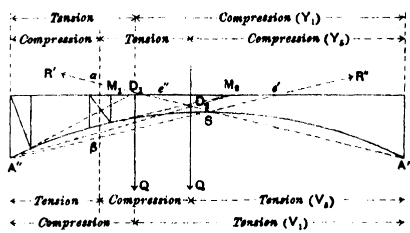


Fig. 175.

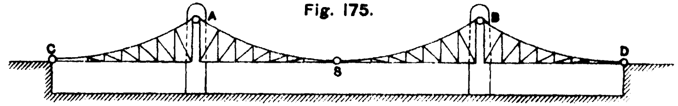


Fig. 176.

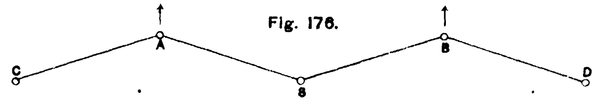


Fig. 177.

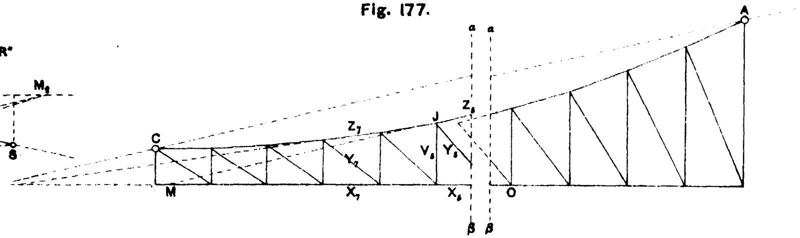


Fig. 178.

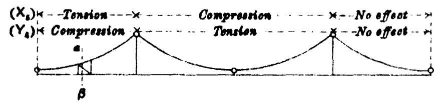


Fig. 179.

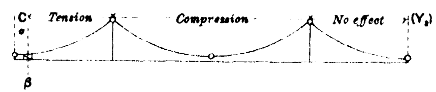


Fig. 180.

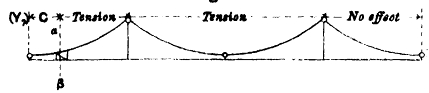


Fig. 181.

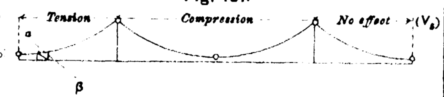
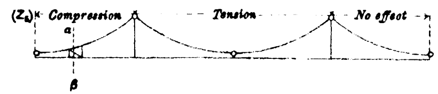


Fig. 182a.



Litho., T. O. Press, Roorkee.

Fig. 183.

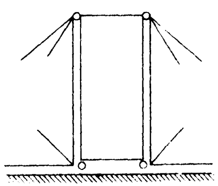


Fig. 184.

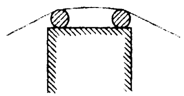


Fig. 185.

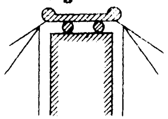


Fig. 186.

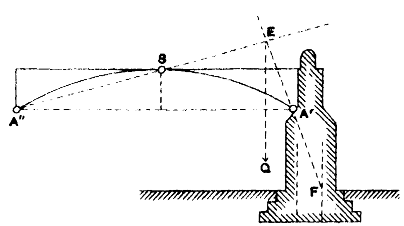


Fig. 192.

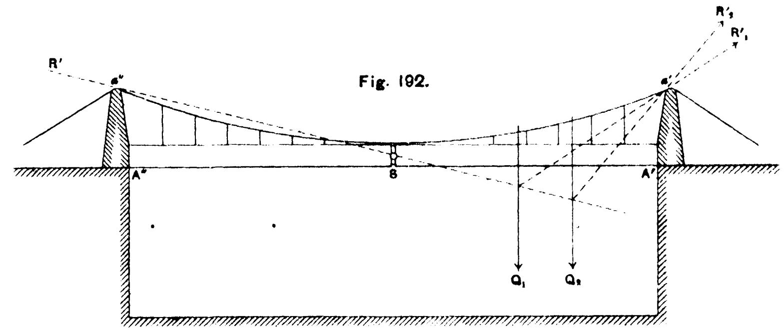


Fig. 187.

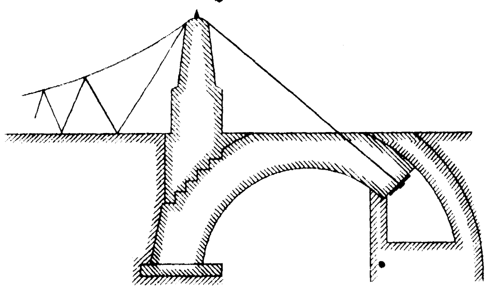


Fig. 188.

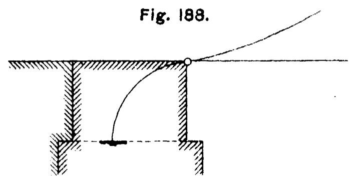


Fig. 199.

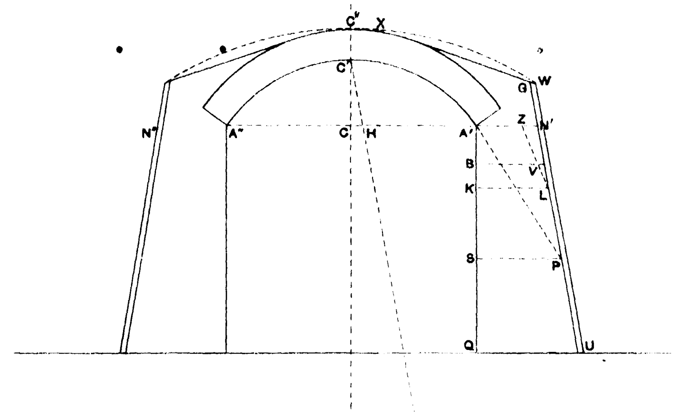


Fig. 189.

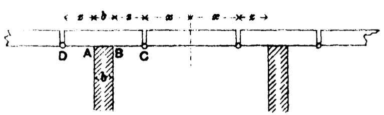


Fig. 190.

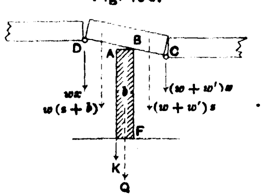


Fig. 191.

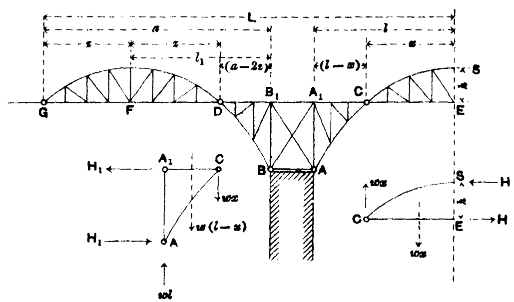


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Fig. 194

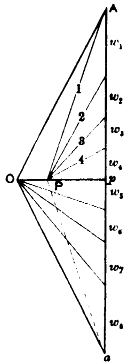


Fig. 193

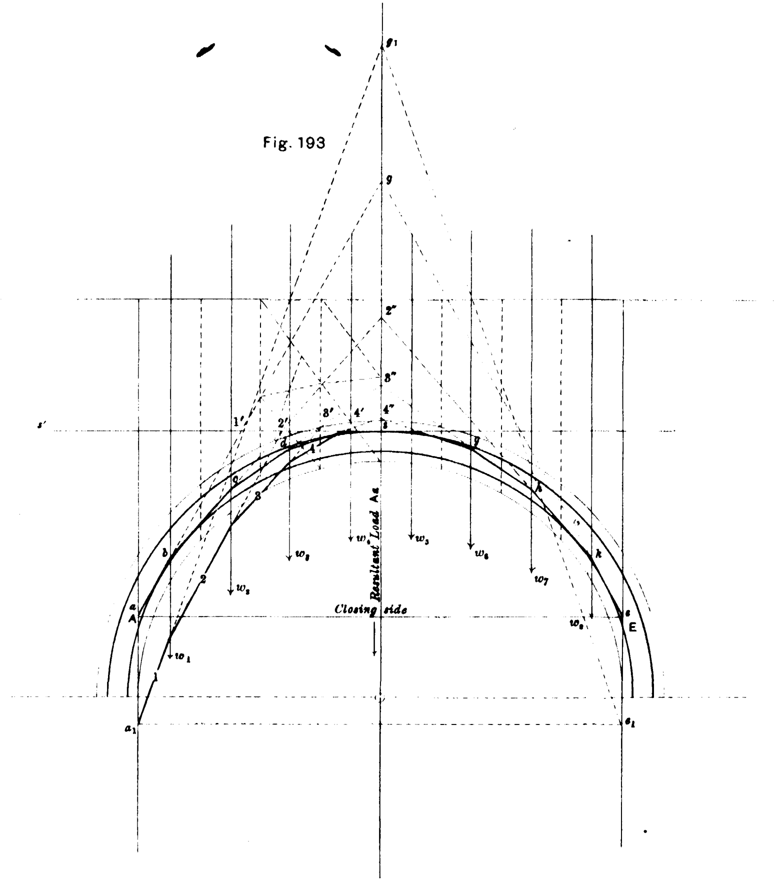
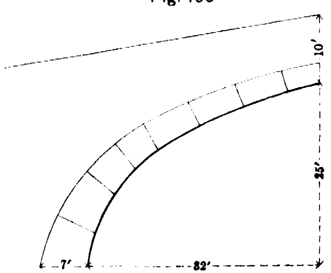


Fig. 195



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bolt as well as at right angles to it. As the thread is usually V-shaped and cut into the bolt, the effective section is that of the sectional area at the root of the thread. The thread or screw on the bolt is called the "male," that on the nut, or sheath in which it works, the "female" thread or screw.

436. If d be the diameter of the bolt before the thread is cut and D that at root of thread, then the ratio $\frac{D}{d}$ depends on the pitch and kind of thread. The threads usually employed in English practice are Whitworth's, and for these

$$D = 0{\cdot}9\,d - 0{\cdot}06$$
$$d = 1{\cdot}1\,D + 0{\cdot}07$$

If l be the length of the "female" screw, or depth of nut, the shearing area against which the screw acts is $\pi \times D \times l$, and an expression can be found for l in terms of s_t and s_s. For, it is obvious that the total tension on the bolt must not exceed the total shearing resistance. Hence, in the limit we have

$$\tfrac{1}{4} \times \pi \times D^2 \times s_t = \pi \times D \times l \times s_s$$
$$\therefore l = \frac{D}{4} \times \frac{s_t}{s_s}.$$

For s_s a low value should be taken as the metal has been weakened by cutting,* say 2 tons per square inch *at most;* for s_t, 5 tons per square inch may be taken.

Hence, $l = 0{\cdot}6\,D$ (*at least*) $= 0{\cdot}54\,d - 0{\cdot}04.$

The common practice is to make $l = d$, which is ample. The width of nuts and bolt heads may be taken at $1\frac{1}{2}$ times the diameter of the bolt.

* Humber's Iron Bridges, p. 151.

CHAPTER XXV.

MASONRY ARCHES.

437. In passing from the consideration of Braced Arches of Iron and Steel to that of Masonry Arches, we leave structures whose joints are of the first class, as defined in paragraph 6, and pass to those having joints of the second, and it becomes necessary to take account therefore not of lines of resistance only, but of lines of *least* resistance. The question of joints in their relation to lines of resistance has been fully dealt with in Chapter V., and likewise that of the resistance at a plane joint in Chapter XI. of this Volume, so that, before passing directly to our subject, we shall merely remind the Student that Masonry Arches are designed as structures of Uncemented Blockwork on the following principles:—

As Uncemented Masonry is incapable of resisting a tensile stress, but can sustain a compressive one within certain limits, the weight to be carried by the Arch is supported on a series of wedge-shaped voussoirs forming the arch-ring, which touch each other and are so disposed as to be eminently fitted to resist compressive stress; for, when loaded by the superstructure, they press against one another and practically form a solid curved mass, within the centre half or third of the thickness of which the line of resistance must be retained under all possible distributions of the load.

438. The two Conditions of Stability that must be fulfilled by Structures of Uncemented Blockwork, as stated in para. 172 of Chapter X., are as follows:—

I. *The line of resistance must intersect every bed joint sufficiently far within the outer edges to prevent any risk of their crushing.*

II. *The angle which the resultant pressure on any bed joint makes with a normal at that joint must not exceed $\frac{4}{5}$ths the angle of repose.*

Condition I., being the condition of *strength*, must be fulfilled for *every hypothetical joint of the structure*, but Condition II., being the condition

preventing the sliding of one voussoir on another, *applies to bed joints only, i.e.*, to the contiguous masonry surfaces of the voussoirs (para. 162); the line of resistance, therefore, must be drawn first so as to fulfil Condition I., a system of hypothetical joints being chosen convenient for estimating the loading, and the compliance, or not, of the voussoir surfaces with Condition II. afterwards ascertained.

To estimate the Load.

439. The first thing to do is to estimate the load. In the bridges that we have hitherto considered, it has been sufficient to assume that each joint is equally loaded both with permanent and moving load, for although not true as regards the superstructure of many of the forms of bridges dealt with, the assumption has been sufficiently accurate for all practical purposes. But in the case of masonry arches the error that would be entailed by such an assumption would obviously be too great, and it is, therefore, necessary to make a more exact measurement of the weight of the superstructure, permanently borne by the structure.

If a right section of the arch and superstructure be drawn, it is evident that, if these be built of the same material, a graphic representation is thus obtained of the permanent vertical load to be dealt with; if they be of different materials, then the equivalent load in terms of some one of the materials being determined and the corresponding section drawn, an equivalent graphic representation is obtained.

For estimating the vertical load it is convenient to employ hypothetical joints dividing the structure evenly and vertically, thus cutting it into strips by a series of vertical planes at equal distances apart, as shown by dotted lines in *Plates* XXXV. and XXXVI.; if the section be considered to be of unit thickness, the weight of each strip can be estimated and supposed concentrated at its centre of gravity.

If the vertical planes be taken sufficiently near together, *i.e.*, if the width of the strips be sufficiently small, each strip may be considered a trapezoid, or, at least, be reduced to one of approximately equal area, and since the width of each trapezoid is the same, the weight of each may be regarded as proportional to the sum of the lengths of its vertical sides, or to any fraction of such combined lengths; and a convenient way of dividing the load line proportionately to the weights of these several trapezoids is the following:—

Lay off from A the vertical component of the *total* load (or, if the load is entirely vertical, then the total load) on the proper scale and in the vertical direction, as Aw_1 in *Fig.* 198, and draw from A *any* straight line Ac in any convenient direction, marking off on it at the points 1, 2, 3, ...14 any convenient fraction of the several combined lengths above referred to, introducing the isolated loads, if any, as W, *Fig.* 198, in their proper places; the vertical line Aw_1 can then be divided proportionately to Ac, by joining w_1 to 14, and drawing through the points 1, 2, 3, ... 13 straight lines parallel to w_1 14.

If the curved sides of the original strips be fairly flat, the positions of their centres of gravity may be considered identical with those of the equivalent trapezoids. This position will be readily found by bisecting each vertical side and joining the points of bisection. The required point *must* lie somewhere in this straight line. Now draw a diagonal and so divide the trapezoid into two triangles. Considering the vertical sides as the bases of these triangles, the middle points of which are already found, the centres of gravity of the triangles are known at once. The centre of gravity of the strips will evidently lie in the straight line joining the centres of gravity of the triangles. Hence its position is thus fixed by intersection, (*vide* para. 90).

If the curved sides of a strip be not fairly flat (as, for instance, in the case of the outer strips of a semi-circular arch), and accuracy is to be observed, the position of the centre of gravity of the strip will be most readily determined by cutting its figure out of cardboard, and finding the required point practically by suspending the same in two positions. The point of intersection of the verticals drawn from each point of suspension is the centre of gravity.

Draw vertical straight lines through the centres of gravity of the several strips to represent the lines of action of the vertical loads; the positions of these loads are fixed.

The moving load can be dealt with as in previous examples; there is nothing particular to note in this respect.

Thickness of the Arch Ring at the Crown.

440. Mr. DuBois, on page 329 of his *Graphical Statics*, points out that the proper depth of arch at the key, or summit of the arch-ring, depends not only upon the rise and span, but also upon the load. The

pressure at the extrados at the key, which is, of course, under ordinary circumstances the most exposed part of the joint, should not, according to the best authorities, exceed one-tenth the ultimate resisting power of the material.

Thus, if p is the pressure per unit of cross section, H the thrust, and d the depth of the key-stone joint, then we have

$$p = \frac{2\,H}{d}$$

the maximum pressure being twice the mean pressure on the usual assumption that the curve of pressure does not pass outside the safe limit. This mean pressure, then, should not exceed $\frac{1}{20}$th the ultimate resistance of the material (since twice the mean pressure must be $>$ $\frac{1}{10}$th of it).

"In general" he says "we must first assume the depth at key in view of the strength of the material, the character of the workmanship, the load, &c. Then, the thrust being found, we find the mean pressure per unit of area as above. If this mean pressure exceeds $\frac{1}{20}$th the ultimate resisting power of the material, make a new supposition, increase the thickness, find the thrust and pressure anew, and so on till the results are satisfactory.

441. It is necessary, in fact, first to make a trial design of the arch, and then test its stability. Rules for such a design are given in all Engineering text-books; the following are those of the American Engineer, Mr. Trautwine, as given in his *Civil Engineer's Pocket Book.* Expressed in algebraic form they are as follows:—

If r be the radius of the arch in feet, such as will give a curve passing through the springings and crown of intrados,
s, the span of the arch in feet,
d, the depth of the key-stone in feet.

Then, for first class cut-stone arches, whether circular or elliptic,

$$d = \frac{\sqrt{r + \frac{s}{2}}}{4} + 0{\cdot}2 \text{ ....(1)}.$$

For second class work this depth should be increased about one-eighth; or for brick or fair rubble about one-third, if the span exceeds 15 or 20 feet.

To find the radius in terms of the span and rise a, we have

$$r = \frac{\left(\frac{s}{2}\right)^2 + a^2}{2\,a} \text{ .... (2)}.$$

The above rules are based upon drawings and calculations made of arches from 1 to 300 feet span, and of every rise. Arch rings should, as a rule, be made deep rather than shallow both for reasons of stability and also for appearance sake.

442. As regards the thickness of abutments, Mr. Trautwine gives the following rules, applicable to the smallest culvert and largest bridge, and to any conditions of abutment and of load. "It gives a thickness of abutment which, without any backing of earth, is safe in itself, and in all cases, against the pressure when the bridge is unloaded." In the case of small arches, however, it depends on the resistance of the earth behind the abutments to an extent increasing as the span diminishes, and in practice, earth is always deposited behind the abutments as the work proceeds.

If T be the thickness of abutment in feet (measured at the springing) when the height does not exceed $1\frac{1}{2}$ times the base,

$$T = \frac{r}{5} + \frac{a}{10} + 2 \text{ ....} (3).$$

If the abutment be of rough rubble, add 6 inches to secure full thickness in every part. In large culverts and small bridges of first class railroads, subject to the jarring of heavy trains at high speeds, abutments from one-fourth to one-half thicker than the above have been employed.

The trial section may be set out as follows:—

Let A′A″, *Fig.* 199, *Plate* XXXIV., be the span with rise CC′ and calculated thickness of arch ring at crown equal to C′C″, by equation (1). Calculate and set out the abutment thicknesses A′N′ and A″N″, by equation (3). From C lay off CH $= \frac{1}{24}$ A′A″ and join C′H, then from N′ draw the indefinite line GN′P parallel to C′H. Make N′G $= \frac{1}{2}$ CC″, and from G draw a line touching the extrados of the arch at X. Then GX is the top of the masonry filling above the arch. From A′ draw A′P sloping at $\frac{3}{2}$, and from the point P in which it cuts the line GN′P draw PS horizontal. If PS is on the ground line, nothing further need be done, but if the ground line is at Q, make the base QU = SP + $\frac{1}{4}$ SQ, and draw the back UW parallel to GP. The line UW now becomes the back of the abutment. "The additional thickness has reference rather to the pressure of the earth behind the abutment than to the thrust of the arch. In a very high abutment the inner line GP would give a thickness too slight to sustain this earth safely."

The width of base, further, should never fall short of one-half the total height from base to springing. If the above rule give a base which is less than half this height, the width must be increased accordingly, and the back, &c., drawn in as before. When, on the other hand, the height of the abutment is less than the calculated thickness, as found by equation (3), a slight reduction may be allowed as follows:—Make A′K equal to A′N′ and draw KL horizontally to meet GP in L. Make A′Z = $\frac{3}{4}$ A′N′ and draw LZ. Then for any height of abutment less than A′N′ (*i.e.*, A′K) draw BV, terminating in LZ. Then the length of BV will be a sufficient base if the foundations are firm. The back of the abutment must be drawn through V parallel to GP and terminating above at the same height as G or W.

443. Professor Rankine's rule for the thickness at the crown, derived from experiments on numerous existing bridges, is as follows:—

For the depth of the keystone, take a mean proportional between the radius of curvature of the intrados at the crown and a constant whose values are:—

For a single arch, ...	0·12
For an arch, forming one of a series, ...	0·17

or, putting it in the form of an equation—

Depth of keystone for a single arch, *in feet* $= \sqrt{0{\cdot}12 + \text{radius at crown}}$.

Depth of keystone for an arch of a series, *in feet* $= \sqrt{0{\cdot}17 \times \text{radius at crown}}$.

The depth at the springing in stone arches is generally more than that at the crown, the pressure being far greater there; it is generally determined in the first instance by considerations of appearance, and tested afterwards for stability and strength.

444. The trial design, having been completed in the manner above described, should be tested for stability by drawing in the line of least resistance, and examining whether the two conditions referred to in para. 438, are properly satisfied.

To draw the Lines of Least Resistance for a given Masonry Arch-ring and a given System of Loads.

The general Case of the Arch is considered in para. 83 of Chapter VI.; each of the three simpler Cases, referred to in para. 73 of the same Chapter, will now be considered; they are the following, and will be found to include all those of ordinary occurrence:—

Case A.—*Where the load is symmetrically situated with regard to the arch-ring and vertical.* This is the most commonly occurring Case of all three.

Case B.—*Where the load is unsymmetrically situated with regard to the arch-ring and vertical.* This Case is one of theoretical rather than practical interest.

Case C.—*Where the vertical load is symmetrically or unsymmetrically situated, and the arch is subjected, in addition, to the pressure of earth, the direction of which may be either horizontal or inclined.* This Case includes examples of tunnels, underground cellars, &c.

Cases A and B are examples of parallel loading; Case C of oblique loading.

To draw the Line of Least Resistance for Cases A, B *and* C.

Case A.—The numerical value of the shearing force at each extreme vertical section (corresponding to the points of support of a horizontal beam) is evidently equal to half the applied load in Case A, and its value zero at the middle vertical section of the arch. Moreover, the direction of the axis of abscissæ of the curve of least resistance is horizontal.

Draw a vertical straight line Aa, therefore, (*Plate* XXXV., *Fig.* 194) to represent the applied load on any convenient scale of loads, and divide it proportionately to the weights of the several trapezoids as explained. Draw a horizontal straight line Pp through its middle point p as locus of poles. Assuming any point P on this locus as pole, describe a stress diagram PAp, and the corresponding resistance polygon a_1 1, 2, 3, 4, 5, *Fig.* 193, for half the arch (since the load is symmetrical), commencing the latter at the middle point s of the upper limiting curve. The side at this point will be horizontal, since the number of strips is even, and will be a tangent to the said curve. Produce this tangent (s' s''), and produce all the other sides of the polygon to meet it in the points 1′, 2′, 3′, 4′. Then project the polygon by revolution round this tangent, as follows :—

From the point 1′ in $s's''$ draw the straight line g1′b to touch the lower limiting curve to the left of the line of action of weight w_1 and to meet that line of action in b. Join the points b and 2′, and let b2′ meet the line of action of w_2 in the point c. Join c and 3′, cutting the line of

action of w_3 in d. Join d and $4'$. The polygon $abcd4's$ represents the left half of the polygon of least resistance, and the pole O may be found by drawing through A (*Fig.* 194) a straight line AO parallel to $b1'g$ to meet the horizontal drawn through p in O. The right-half-polygon may be described either by means of pole O in the ordinary way, or by producing the sides of the left-half-polygon to meet the line of action of total resultant load gs in the points g, $2''$, $3''$, $4''$, and joining g to e, the right extremity of the closing side ae, cutting line of action of weight w_8 in k; then joining k to $2''$ cutting line of action of w_7 in h; joining h to $3''$ cutting line of action of w_6 in g, and so on. The polygon $abcd......ke$ enables the curve of least resistance to be drawn in.

Case B.—In Case B draw any complete closed resistance curve, as in *Fig.* 198, *Plate* XXXVI., and determine the position of the section of greatest bending moment, and the values of the shearing stresses at the extreme vertical sections. Now reduce the ordinates of the limiting ring curves, as measured from the axis of abscissæ AE in the ratio of Gg (*Fig.* 196) to $g'g''$ (*Fig.* 198), and plot these reduced lengths in their proper positions from the axis ae, (*Fig.* 198). It is now evident on inspection that the critical sides are 3 or 4, 12 and 14, and that the line of least resistance will touch the upper limiting curve somewhere near the point of contact of side 12. Assume a point of contact at t (*Fig.* 198), and draw the vertical section $t't''$ through t. Let $t't''$ produced intersect the upper limiting curve (unprojected), *Fig.* 197, at T. Draw the tangent SS at T, and project it on to the polygon, *Fig.* 198, by reducing its ordinates as measured from A and E (*Fig.* 197), in the ratio above mentioned, *i.e.*, in that of Gg (*Fig.* 196) to $g'g''$ (*Fig.* 198), thus obtaining the tangent ss to the curve of resistance, *Fig.* 198. Produce the sides of the polygon (*Fig.* 198) to meet the tangent ss in the points 1, 2, 3, and project these points, by vertical lines, on to the tangent SS. Now describe a new polygon as $a_1g_1e_1$, *Fig.* 197, such that the directions of its sides 1, 2, 3, 4, &c., shall pass through the respective points 1, 2, 3, 4, &c., of SS, and those of its extreme sides 1 and 15 shall meet on the plane of the vertical section of greatest bending moment as at g_1', and such that all its sides shall lie as far as possible within prescribed limits and *yield as concave a corresponding curve as possible.*

Now produce the sides of this polygon to meet the section $t't''$ in the points 1, 2, 3, 4, &c.,......and reproject the polygon as before (*viz.*, by revolution round the ordinate $t't''$ produced), so as to fulfil the requirements already detailed, thus obtaining the polygon $a_2g_2e_2$. In doing this it must be remembered that the corresponding sides of the polygons $a_1g_1e_1$ and $a_2g_2e_2$ produced meet in $t't''$, that the ordinate Tt (*Fig.* 197) remains unchanged, and that the directions of the extreme sides 1 and 15 meet on the plane of the section of greatest bending moment, as at g_2'. The polygon $a_2g_2e_2$, thus obtained will correspond to the line of least resistance, and its pole P may therefore be found, and the values of the thrusts on the abutments, *viz.*, PA and Pa (*Fig.* 198) be measured. In *Fig.* 196 the line of least resistance $a\ g\ e$ has been described with pole P.

Case C.—The vertical loads are first determined, as already explained for cases A and B, by supposing the structure to be composed of vertical strips, all of the same breadth. The several conjugate earth pressures acting on the strips are then calculated or determined graphically. These two systems of loads are represented on the diagram of external loads, and being combined, constitute a system of oblique loading, which may be dealt with in the manner already explained in para. 83.

Examples.

Case A.—Examine the stability of the symmetrical and symmetrically loaded arch, half of which is shown in *Fig.* 198, *Plate* XXXVI., span 64 feet, rise 25 feet; the material of the arch-ring is the same as that of the superstructure it carries, which is solid across the span, and weighs 1 cwt. per cubic foot.

Case B.—Design the ring and abutments and examine the stability of a symmetrical segmental arch of 40 feet span, 10 feet rise, and 25 feet radius, the distance from intrados to top of superstructure measured at crown being 5 feet; the material of the superstructure (which is solid across the arch) and ring is the same and weighs 1 cwt. per cubic foot; an additional uniform load of 2·5 tons per foot run extends from the left abutment up to 15 feet across the span.

Domes and Tunnels.

445. The principles on which the stability of hemispherical and

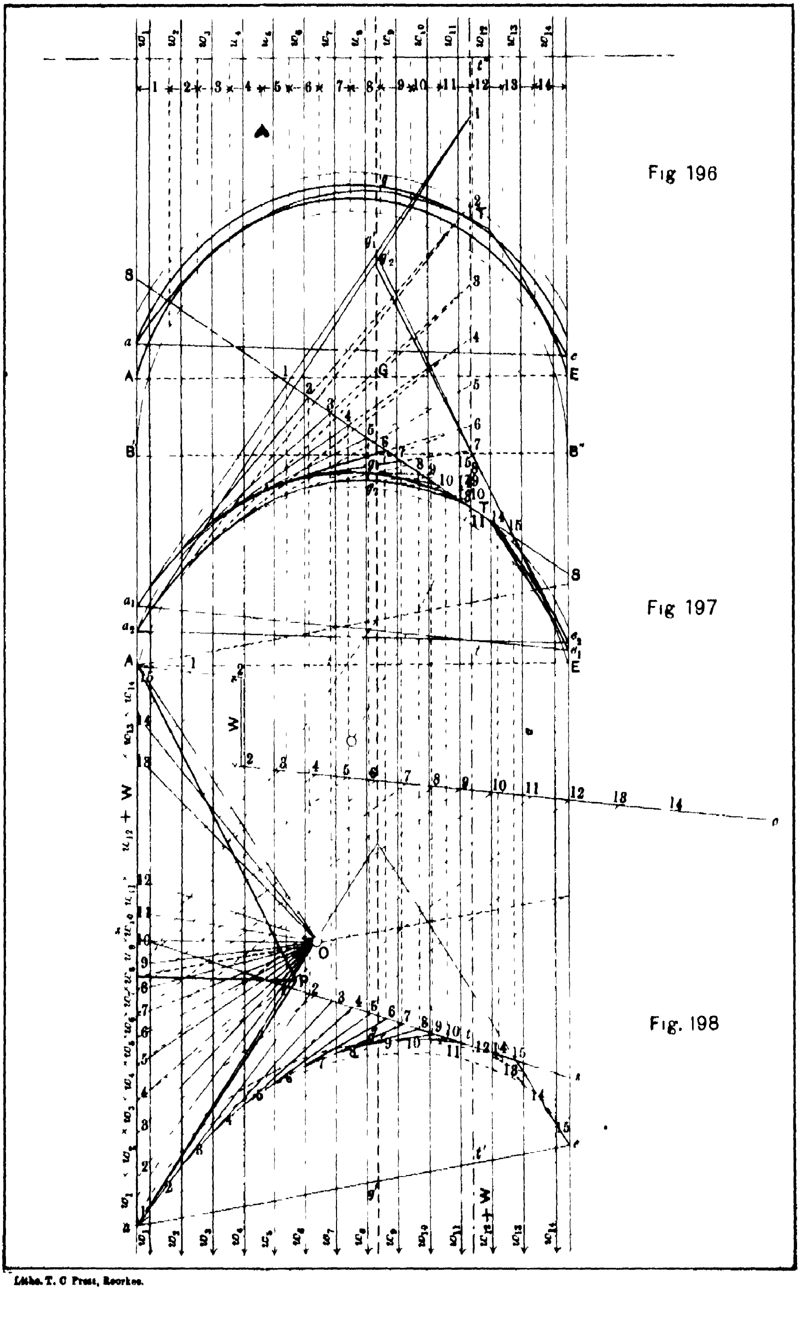

Fig 196

Fig 197

Fig. 198

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Fig. 200.

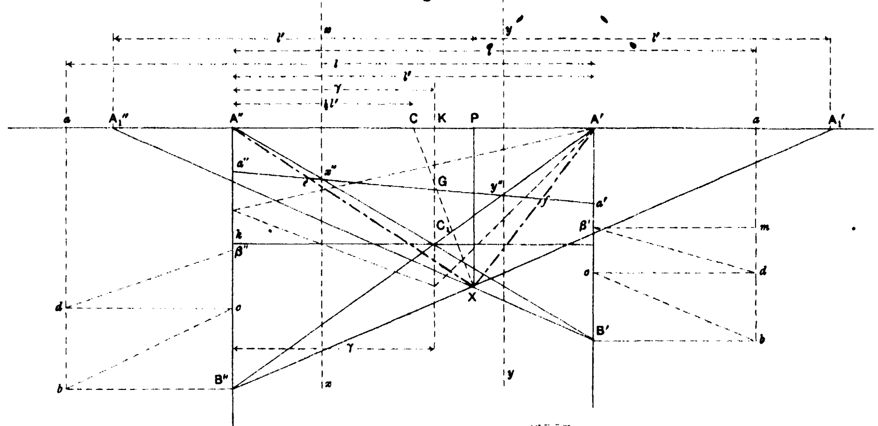


Fig 201.

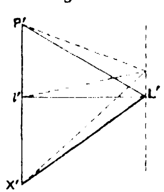


Fig. 202.

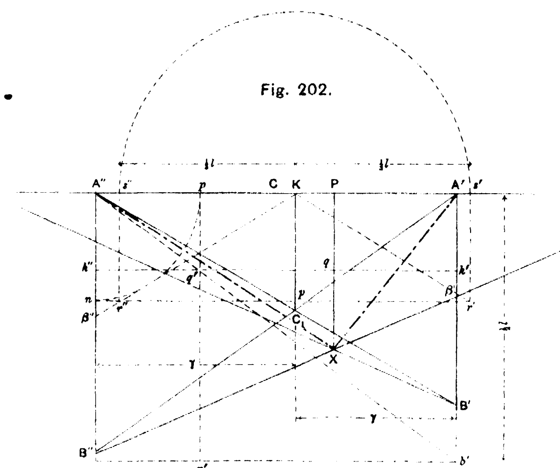
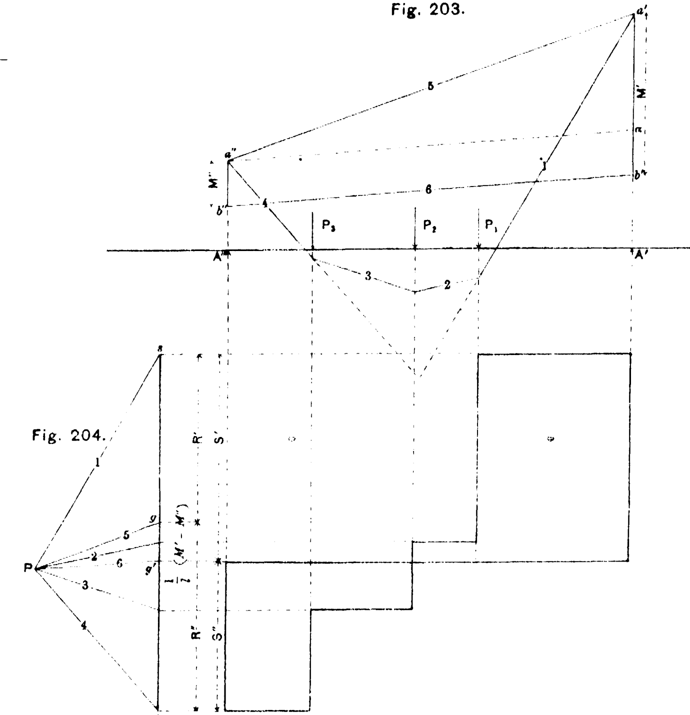


Fig. 203.

Fig. 204.



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spheroidal domes of masonry is examined are the same as those above explained for masonry arches, the dome being divided by a series of vertical planes containing the axis of the sphere or spheroid, and also by cylindrical planes, having a common axis identical with the vertical axis of the dome. The segment of the dome to be dealt with is considered with reference to the vertical plane dividing it symmetrically, and it will be observed that the magnitudes of the loads of the several parts of this segment, so divided, increase at first rapidly from crown to springing, each representing the weight of a certain portion of an annulus of the dome. If the pressure of the wind be taken into account, the loading on one side of the dome becomes oblique, and the method to be employed is that of Case C of para. 444.

The examination of the stability of tunnels and underground cellars falls, as already intimated, to Case C of the previous paragraph.

CHAPTER XXVI.

FIXED AND CONTINUOUS GIRDERS.

446. In Chapter VIII. the general application of the equilibrium polygon to the measurement of the deflection of uniform beams is explained, and in Chapter IX. the method is extended to the measurement of the stability of fixed and continuous beams. The treatment of the two cases was shown to be similar, a difference arising, however, out of the existence in the latter classes of beams of certain bending moments at the points of fixation or intermediate support as the case may be, the values of which are arrived at by means of deflection polygons drawn for the given conditions of loading so as to occupy a particular position in regard to certain so-called "fixed" points. These Bridges we shall now more fully discuss, including, as they do and as has been already explained, Cantilever Bridges, which are a natural and obvious development of the former type.

447. But before doing so, it will perhaps be as well to say a word in regard to the representation of moments by straight lines. "The moment of any physical agency" has been defined to be "the numerical measure of its importance." "The moment of a force round a point or line signifies the measure of its importance as regards producing or balancing rotation round that point or round that line."* It takes into account not only the magnitude and direction of the force, but also its distance of operation from the point or line in question, and measures the tendency to produce rotation either one way or the other. The direction of the hands of a watch is usually regarded as positive and the reverse direction as negative, but this is purely conventional, and when, therefore, the bending moments of the directly applied forces at any cross section of a continuous or fixed girder are plotted below and those of the indirectly applied forces above a given datum, the meaning simply is that at this cross section the tendencies to produce rotation of these

* Thomson and Tait's *Elements of Natural Philosophy.*

respective forces are in opposite directions, the numerical value of the resultant moment being measured by the difference of those of the component moments, and the resultant tendency being, as a rule, obvious.

448. It will be well to recapitulate some of the principal points established in Chapter IX.

In paras. 126 to 132, it is shown that the system of loads *directly* applied in the span itself gives rise to equilibrium polygons which are precisely similar to those that would obtain were the beam discontinuous at the several points of support and freely supported, and that this system of moments can be regarded quite independently of any due to loads *indirectly* applied and acting outside of the span in question.

In paras. 144, 145 it is shown that the loci of the "fixed" points are vertical straight lines which occupy a certain definite position within the outer third of each span quite independently of any loading directly applied within that span.

In para. 155 it is shown that in a series of adjacent and unloaded spans the moments at the supports must necessarily be represented as acting alternately downwards and upwards as we pass away from the loaded span, while in loaded spans they must invariably be shown as acting upwards, (*see Plate* XXXIX.)

449. It follows from the above that in a series of unloaded spans of a continuous girder, only one span of which, suppose, is loaded, the deflection polygon must cut the line joining the points of support once only in each unloaded span at an inflexion point of zero moment, and that this point lies on that one of the two "fixed" verticals of that unloaded span which is further away from the loaded span; also that the values of the moments over the points of support of this series of unloaded spans diminish gradually from the loaded span outwards, each being less than one-third of that which is next nearer to the loaded span, for the obvious reason that the ratio of the magnitudes of these moments must be the same as that of the segments into which the span is itself divided by the deflection polygon, that is, by the inflexion point, (para. 135.)

450. In the case of a loaded span, obviously the deflection polygon crosses the line joining the supports twice in each span, that is, once in each outer third of the span.

It will, moreover, be perceived, as already explained, that cases of continuous loading may always be regarded as merely special cases of detached loading; for in any case the area of the curve or polygon of bending moments has to be reduced to a rectangle with base of half the actual or standard span, that is, to an equivalent detached load, before the deflection polygon can be drawn in.

451. It will thus be observed that there is no limit whatever to the nature of the loading which may be dealt with by this geometrical method, and it should be noted in this connection that the analytical method treated of in Chapter XVII., Vol. I., only includes cases of uniform beams symmetrically loaded.

With the help of these remarks, we shall proceed to a further brief consideration of the stability of continuous and fixed beams.

Continuous Girder of several spans, with only one span loaded.

452. In paras. 138 to 140, it is explained that when the several spans of a continuous girder are of unequal length, the first thing to do is to measure for each span the length of the intercept made by the extreme sides of the deflection polygon on certain selected vertical lines, generally those drawn either through the "fixed" points or the supports, and then, before the deflection polygon for the whole girder can be described, to reduce these intercepts to what they would be were each span the same length as the standard one, so that one-third that length may be employed for the pole distance of the force polygon of the deflection and elastic polygons.

453. The method of describing the deflection polygon is explained in para. 156, and from it the elastic polygon is deduced on the principle stated in para. 135, *viz.*, that whereas in the case of the former polygon the *full* component moment areas are employed, in that of the latter, the *resultant* moment areas only are so. The elastic polygon having been described, the elastic curve can be drawn in and the maximum deflection determined by means of para. 119. This is more fully explained in para. 461.

Moments over Supports and Intercepts on chosen verticals.

454. In the method described in Chapter IX., the several spans of the girder whether loaded or unloaded are dealt with simultaneously,

but it is often convenient, especially when examining what dispositions of the loads on the several spans will produce maximum strains (as will be subsequently explained), to deal with each one separately on the hypothesis that it alone is for the time being loaded and all the others unloaded. The positive and negative moments over each support being then added and their resultants plotted, a diagram of maximum resultant moments is obtained, (*Fig.* 212).

455. The following is an extremely simple and convenient method (due to Professor Lippich) of setting off the intercepts on the verticals through the supports, made by the sides of the deflection polygon which support the resultant applied load, enabling the moments over supports to be at once determined :—

Let $A''XA'$, *Fig.* 200, represent the resultant bending moment area of a system of loads hung between the points of support A'' and A'. It is required to measure the intercepts at the supports of the deflection polygon.

Construction.—Set off PA_1' and PA_1'' on either side of P, (the upper extremity of the maximum ordinate of $A''XA'$) each being equal in length to the span $A''A'$. Join $A_1'X$ and $A_1''X$ and produce them to meet the verticals through the supports A'' and A' respectively in the points B'' and B'. Then will $A''B''$ measure the intercept on the vertical through A'' and $A'B'$ that on the vertical through A', on the same scale as PX measures the maximum bending moment at P.

Proof.—The polygon of bending moments being represented by the triangle $A''XA'$, let the intercepts $A''B''$ and $A'B'$ be found in the usual way by means of a second force polygon whose load line $P'X'$ is equal to PX and pole distance $l'L'$ to $\frac{1}{3}$ $A'A''$. This force polygon is shown in *Fig.* 201 by the triangle $P'L'X'$ (the trial force and corresponding equilibrium polygons being both shown in dotted lines). The resulting equilibrium polygon $A''C_1A'$ is the deflection polygon of the downward, or imposed, loads, and consequently the vertical C_1K, drawn through its apex C_1, passes through the centre of gravity G of the triangle of bending moments $A''XA'$. It will further be observed that the triangle $A''C_1B''$ of *Fig.* 200 is similar to the force polygon $P'L'X'$ of *Fig.* 201. Join $B'X$ and $B''X$, and produce them to meet $A'A''$ produced both ways in A_1'' and A_1' respectively. Then will PA_1' and PA_1'' be each equal to $A'A''$, the span length.

For, since PX is parallel to $A''B''$

$$\therefore \frac{A''B''}{PX} = \frac{A''A_1'}{PA_1'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and comparing the similar triangles $P'L'X'$, *Fig.* 201 and $A''C_1B''$, *Fig.* 200, we have

$$\frac{l'L'}{P'X'} = \frac{C_1k}{A''B''} \text{ or } \frac{A''B''}{P'X'} = \frac{C_1k}{l'L'}$$

But (by hypothesis) $l'L' = \frac{1}{3} A'A''$ and $C_1k = A''K$

$$\therefore \frac{A''B''}{P'X'} = \frac{3\ A''K}{A'A''}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence, from (1) and (2), remembering that $PX = P'X'$ we have

$$\frac{A''A_1'}{PA_1'} = \frac{3\ A''K}{A'A''}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

But, since KC_1 passes through the centre of gravity of $A''XA'$, if $A''A'$ be bisected in C and CX be joined cutting KC_1 in G then is G the centre of gravity of $A''XA'$, and, therefore, $PK = 2\ KC$ and $A''K = A''P - \frac{2}{3}\ PC$, so that $3\ A''K = 3\ A''P - 2\ PC$.

But $PC = PA'' - A''C = PA'' - \frac{1}{2}\ A''A'$

$$\therefore 3\ A''K = A''A' + A''P, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Hence from (3) and (4) we have—

$$\frac{A''A_1'}{PA_1'} = \frac{A''A' + A''P}{A''A'}$$

Subtracting numerator and denominator, we have—

$$\frac{A''P}{PA_1'} = \frac{A''P}{A''A'}, \text{ whence } PA_1' = A''A'.$$

In a similar way it may be shewn that $PA_1'' = A''A'$.

456. This method is directly applicable to the determination of the moments over the points of support, when the "fixed" points fall on the straight line joining the latter, as would be the case under the following conditions:—

Case I.—Were the span under consideration the only one of the series loaded, all the others being unloaded; because we should, then, in order to obtain the lengths $A''a''$ and $A'a'$, (representing the moments over the respective supports A'' and A'), simply have to join the points x'' and y'', *Fig.* 200, in which the "fixed" verticals x and y cut the deflection polygon $A''C_1A'$ and produce the straight line $x''y''$ to meet the verticals through the supports A'' and A' in the points a'' and a' respectively.

Case II.—When the beam is "fixed" at both ends, because we have then only to draw through x_1 and y_1, *Figs.* 217 and 222 (the points in which the verticals drawn at one-third the span length from either support cut the deflection polygon) a straight line, produced to meet the verticals through the supports, in order to find the moments $A''a''$ and $A'a'$ over the supports. This will be evident after the inspection of any deflection polygon.

Examples are given of each of these cases at the end of this Chapter.

Reduction of Intercepts on Verticals through Supports.

457. The intercepts on the verticals through supports, which have been determined for the actual span under consideration, either by the method of Chapter IX., or that above described, must now be reduced to standard lengths. We shall deal with a single resultant load, giving a triangular moment area, which latter may, as already explained, be the equivalent of a curve or polygon.

If l be the length of the standard span, l' that of the span under consideration, y' that of the longest ordinate of the moment area of span l' (in this case, the height of the moment triangle), then the area of the moment triangle, whose base is l' and height y' is $\frac{1}{2}\,l'y'$, and the height y, therefore, of a triangle of equal area, having base l, must be such that $y = y'\,\frac{l'}{l}$.

Now from *Figs.* 200 and 201 and para. 140, it is evident that, since the triangles $A''C_1B''$ and $P'L'X'$ are similar, we have for the intercept $A''B''$ on the vertical through A'' the relation

$$A''B'' : P'X' :: A''K : l'L'.$$

Now $l'L'$ of *Fig.* 201 is the pole distance of the second force polygon and equals one-third the actual span length under consideration, and $P'X'$ is the load line of that force polygon and is equal to the height of the moment triangle, so that, putting i' for $A''B''$ and γ for $A''K$, the relation becomes

$$i' : y' :: \gamma : \frac{l'}{3}, \text{ whence } i' = 3\,\gamma\,y'\left(\frac{1}{l'}\right), \ldots\ldots\ldots\ldots (\alpha).$$

If now, instead of $l'L'$ we are to write one-third the standard span length, for $P'X'$ we must substitute the reduced value y of y' as above obtained, or $y'\,\frac{l'}{l}$, and the relation then becomes, putting i for i',

$$i : y' \frac{l'}{l} :: \gamma : \frac{l'}{3}, \text{ whence } i = 3\,\gamma y' \frac{l'}{(l)^2}, \ldots\ldots\ldots\ldots\ldots \quad (\beta).$$

458. The reductions above described admit of easy graphical treatment.

Thus, for the reduction of the height of the triangle—Let A″XA′, *Fig.* 202, represent the resultant triangle of bending moments. On the vertical through A′ set off A′b' downwards and equal to half the length of the standard span, and complete the parallelogram A″b'. Bisect PX in q, and through q draw $k''qk'$ parallel to A″A′ and cutting the verticals drawn through A″ and A′ respectively in k'' and k'. Then the rectangle A″k' is equal to the triangle A″XA′. Draw the diagonal A″b' cutting $k''k'$ in q', and through q' draw $pq'p'$ parallel to A″B″ meeting A″A′ in p. Then obviously the rectangle A″p' is equal to the rectangle A″k', since the complement $k''p'$ is equal to the complement q'A′; that is, the rectangle A″p' is equal to the triangle A″XA′. Therefore the distance of pp' from A″B″ measures the height of the triangle whose base is equal to the length of the standard span and area equal to that of A″XA′.

The intercepts may be reduced from this as follows:—

From equation (β) we have (*Fig.* 202):—

Required intercept : A″p :: A″K : $\frac{1}{3}$ standard span.

Lay off, then, A″n = A″p towards B″, and from K lay off Ks'' = $\frac{1}{3}$ standard span towards A″. Through n draw nr' parallel to A′A″ and through s'' draw $s''r''$ parallel to A″B″ meeting nr' in r''. Join Kt and produce to meet A″B″ in β''. Then A″β'' is obviously the required reduced intercept at A″.

In a similar way the reduced intercept A′β' may be found at A′, and these lengths will be found to be the same as those found by the method described in the following paragraph.

459. The intercepts may, however, be directly and more shortly reduced as follows:—

From equation (α) above we have $i' \times l' = 3\gamma y'$; that is, referring to *Fig.* 200, A′B′ × A″A′ = $3\gamma y'$. If then, from A″ the length A″a be laid off along A″A′ and *towards* A′ equal to that of the standard span l, and the parallelogram A″b be described on it, and the diagonal A″b be drawn cutting A′ B′ produced if necessary in c, and through c the straight line dc be drawn parallel to A′A″, then

will the rectangle A″d be equal to the rectangle A″B′, that is, we shall have

$$ad \times l = \text{A}'\text{B}' \times \text{A}''\text{A}', \text{ whence } ad \text{ or } \text{A}'c = 3\gamma y' \left(\frac{1}{l}\right).$$

Now join A″d cutting A′c in β', and through β' draw $\beta' m$ parallel to A″A′, then will the rectangle A″m = rectangle A″c, and we shall have

$$am \times l \text{ or } \text{A}'\beta' \times l = \text{A}'c \times l', \text{ whence } \text{A}'\beta' = 3\gamma y' \frac{l'}{(l)^2}.$$

In a similar way A″β'' may be deduced from A″B″.

460. In the case of a beam uniformly loaded, the curve of bending moments is a parabola, whose area is equal to ⅔rd that of the corresponding rectangle, so that, using the same notation as before, the area of the curve whose base is l' and greatest ordinate y' is $\frac{2}{3}\, l'y'$, therefore the height of the rectangle of equal area whose base is $\frac{l'}{2}$ must be $\frac{4}{3}\, y'$; and therefore that of the rectangle having the same area and base equal to $\frac{l}{2}$, must be $\frac{4}{3}\, y'\left(\frac{l'}{l}\right)$. This is P′X′ and γ being in this case $\frac{l'}{2}$ and the pole distance of the force polygon, as before, $\frac{l}{3}$ we have (*see* also para. 140) reduced intercept $= f' \frac{4}{3}\, y' \left(\frac{l'}{l}\right) \times \frac{l'}{2} \times \frac{3}{l} = 2y' \left(\frac{l'}{l}\right)^2$.

Deflection and Elastic Polygons.

461. The deflection polygon, described by the method of para. 156 determines the moments over the supports, and from it the elastic polygon, or curve, of the beam is readily deduced on the principle referred to in para. 453.

Thus, *Fig.* 213, *Plate* XXXIX., shows, in chain line, the deflection polygon described for the several unequal spans of the beam A_0A_5, weighted as in *Figs.* 210*a* to *e*. The same polygon for the three bays $\text{A}_0\ \text{A}_3$ is plotted to double the scale and shown, likewise in chain line, in *Fig.* 215. *Fig.* 214 exhibits the *resultant* moment areas of the same three bays plotted to the same double scale, and the elastic polygon, shown in *Fig.* 215 by a firm thick line, may be deduced from the deflection polygon as follows :—

Find the positions of the centres of gravity of the several *resultant* moment areas, shown by small circles in *Fig.* 214, and through them draw vertical straight lines to meet the deflection polygon. From the several points of zero moment t, *Fig.* 214, draw straight lines ts parallel

to the diagonals of the several trapezia of upward moments. We shall then have for any span, as $A_1 A_2$,

$$A_1 s : A_1 a_1 :: A_1 t : A_1 A_2 :: \Delta a_1 A_1 t : \Delta a_1 A_1 A_2,$$

so that $A_1 s$ bears to $A_1 a_1$ the same ratio that the *resultant* upward moment area acting through g does to the upward moment area acting through e (that is, at one-third the span distance from A_1).

Seeing, then, as explained in para. 158, that the triangles $ae\beta$, *Fig.* 215, may be taken to represent the corresponding force polygons of the deflection polygon of the several spans, it would seem to be evident from *Fig.* 215 how the force polygons of the corresponding elastic polygon, (as ebc of the several spans), and so the elastic curve, are deduced (paras. 134 and 135), and no further explanation will therefore be offered.

The force polygon for the elastic curve of span $A_0 A_1$ is shown in *Fig.* 216 to treble the scale of that of *Fig.* 215.

*The Shearing Force in Continuous and Fixed Beams.**

(1). *Loaded Segments.*

462. In paras. 14 and 15 of this Volume it is shown that the reactions at the supports of a freely supported beam are determined by the ratio of the segments into which the vector of the force polygon, which is drawn parallel to the closing side of the equilibrium polygon, divides the total resultant load, and in para. 36 it is explained how the magnitude of the shearing force at any section of a beam may be obtained from the force polygon.

Suppose the closed polygon a' 1 2 3 4 a'' 5 a' of *Fig.* 203 to represent the bending moment area of the beam $A'A''$ which is freely supported at its extremities A' and A'' and loaded with the three detached loads P_1, P_2 and P_3.

If the polygon sPt of *Fig.* 204 represent the corresponding force polygon, then will the vector Pg, drawn parallel to the closing side 5, divide the total resultant resistance ts at g so that tg measures the reaction R'' at A'' and gs that at A', that is R', on the same scale as st measures the total applied load.

Now, suppose on the other hand that $A'A''$ is the line joining the points of support of the segment $A'A''$ of a continuous beam, or the

* *Graphical Determination of Forces in Engineering Structures* (Chalmers'), page 21, *et seq.*

point of fixation of a beam fixed at A′ and A″, the beam being loaded as explained, and suppose that over the support or point of fixation A′ it is subjected to the additional action of the moment M′, represented by $b'a'$, and over A″ to the action of the moment M″, represented by $b''a''$, these moments being due, in the case of the continuous girder, to the action of loads acting outside the span A′A″, and in the case of the Fixed Beam to the material of the wall in which it is fixed.

Then will the resultant bending moment area be represented by the closed polygon $b'a'\,1\,2\,3\,4\,a''b''b'$, and if sPt represent, as before, the force polygon, then will the vector Pg' drawn parallel to the closing side 6 divide the total resultant resistance ts at g' so that tg' represents the reaction S″ at A″ and $g's$ that at A′, that is, S′.

Now, if the moment over the support A′, due to loads indirectly applied, be denoted by M′ and that at A″ by M″, the shearing force at A′ by S′ and that at A″ by S″ and the length between the points A″ and A′ by l, we have, taking moments about A″ and denoting the distance of any weight from A″ by x'',

$$S'l = \Sigma_0^l\, Px'' + M' - M'', \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

moments tending in the direction of the hands of a clock being shown positively—

$$\therefore\; S' = \frac{1}{l}\,\Sigma_0^l\, Px'' + \frac{1}{l}\,(M' - M''), \;\ldots\ldots\ldots\ldots\ldots (2).$$

But $\frac{1}{l}\,\Sigma_0^l\, Px''$ is the reaction at A′ of a discontinuous beam freely supported at A′ and A″, that is, R′.

Hence, $S' = R' + \frac{1}{l}\,(M' - M'')$, so likewise $S'' = R'' + \frac{1}{l}\,(M'' - M')$.

Now, R′ is shown on the force polygon, *Fig.* 204, by gs, and the expression $\frac{1}{l}\,(M' - M'')$ is the portion of the load line intercepted between the lines Pg and Pg'.

For the triangle gPg', *Fig.* 204, is evidently similar to the triangle $a'a''a$, *Fig.* 203—

$$\therefore\; gg' : \text{Pole distance} :: a'a : l :: (M' \backsim M'') : l$$

$$\therefore\; \frac{gg'}{\text{Pole distance}} = \frac{1}{l}\,(M' \backsim M'').$$

But the Pole distance is a reduction base and might be taken equal to unity.

Hence gg' represents $\frac{1}{l}(M' \backsim M'')$ in the same way that any other quantity is represented by a part of the force polygon.

Whence, remembering para. 173, Vol. I., in which it is explained that the shearing force at any section of a beam is measured by the resultant of all the external forces acting on one side of the section, we see that the shearing force at any section of a fixed or continuous beam is obtained in a manner exactly similar to that in which the shearing force of a beam freely supported at its extremities is obtained, once the moments at the supports and the bending moment area are known.

463. From the above it is evident that were the moments over the supports equal, as would be the case *were the Continuous or Fixed Beam symmetrically loaded*, then would M′ be equal to M″ and $a'a''$ be parallel to A′A″ in *Fig.* 203, and Pg would coincide with Pg' in *Fig.* 204, and the magnitudes of the Shearing Forces of Fixed and Continuous beams would, under these circumstances, be the same as those of a similar and similarly loaded discontinuous beam freely supported at its extremities, as stated in para. 310, Vol. I.

(2). *Unloaded Segments.*

464. If the direct loads P in previous investigation be omitted, the case becomes that of an unloaded segment, and we have for the reactions at the supports

$$S' = -S'' = \frac{1}{l}(M' \backsim M'').$$

The shearing forces at the supports constitute, in fact, a couple balancing the moments produced by the loads applied in other spans, and these forces must, therefore, act alternately upwards and downwards at the successive supports (compare para. 155).

Partial Loading of Continuous and Fixed Beams.

465. The case of partial loading should require no further explanation seeing that the partial load when reduced to its single resultant may be treated as a single detached load. The moment area of the partially applied uniform load only would, of course, consist of a parabola joined to a triangle, one side of the latter being a tangent to the former, as shown in *Fig.* 205, *Plate* XXXVIII.

*Conditions for Maximum Positive and Negative Bending Moments and Shearing Forces in a Continuous Girder, subjected to the action of a uniform moving load.**

466. In order to ascertain the maximum stresses to which the material of a Continuous girder is liable to be subjected during the passage of a given uniform moving load, or of loads, it is generally necessary to examine the effect on its parts of every possible distribution of the given load system, and the following considerations are introduced in order to assist the Student in forming an idea of the scheme on which such an examination should be based.

For instance, in a girder of four spans, the sixteen different states of loading shown in *Fig.* 206, in which a thick line denotes the loading, would have to be more or less completely dealt with.

We shall, for simplicity, as before, regard the girder as weightless and deal with the moving load or loads only, forces acting upwards being treated as positive and downwards as negative, and shall divide the investigation into two parts, and consider—I., *A chosen Span Loaded*; and II., *A chosen Span Unloaded.*

I. Chosen Span Loaded.

(1). *Conditions for maximum Shearing Force.*—As in the case of the girder freely supported at its extremities and dealt with in para. 182, Ex. 11, Vol. I., the shearing force is a maximum at any given section when the moving load moving on to the girder from either support extends from that support up to the section in question, being a negative maximum when the right segment of the girder is loaded and a positive maximum when the left segment is so.

(2). *Conditions for Maximum Moment at any given section.*—This depends upon whether the given section falls—(*a*) between the "fixed" verticals, (*b*) between points of support and "fixed" verticals.

(*a*). *Section between "fixed" verticals.*—Since in *Fig.* 200 A_1'' always lies beyond A'' and A_1' beyond A', both A_1'' and A_1' being outside the span, and since $A''B'$ and $A'B''$ fix the positions of x'' and y'', it follows that $A''X$ and XA', also the points of inflection e and f, must always lie *outside* of x'' and y''; consequently, as eXf is the excess of the downward, or negative, over the upward or positive, moment area,

* *Graphical Determination of Forces in Engineering Structures* (Chalmers'), page 232, *et seq.*

it follows that for every section of the girder between x'' and y'', that is, between the "fixed" verticals, every position of a detached load produces a negative or downward bending moment; whence it follows that, *for every cross section of the girder lying between the "fixed" verticals there is a negative moment when the span is fully loaded, that being obviously the condition affording the maximum bending moment.*

(*b*). *Section between points of support and "fixed" verticals.*—Consider any section r between A′ and fixed vertical y, *Fig.* 200. It is evident that if the detached load P, moving on to the girder from the point A″ towards A′, has advanced but an infinitesimal distance from A″, the inflection point f will fall infinitesimally near to the "fixed" vertical y on the line joining the points of support, that is, to y'', because the case will approach infinitesimally nearly to that of an unloaded span; also, that as the load moves further towards A′ so also will the inflection point move towards A′. There is, then, obviously some position π for the load P for which f coincides with r, that is, for which the value of the bending moment at section r is zero; and it is further obvious from what has been said, that as the load passes gradually across the girder from A″ to A′, since the value of the bending moment changes continuously, the latter must have a positive value at section r for all positions of P to the left of π and a negative value for all to the right of π. Hence the following theorem:—

At any cross section r between one of the "fixed" verticals and the nearest support, the bending moment has a positive or negative maximum value, as the case may be, when the continuous load extends from that support to a cross section π of the girder for which, as point of action of a concentrated load, the cross section r is an inflection point, the continuous load being, of course, regarded as a series of very small detached loads.

II. Chosen Span Unloaded.

467. Seeing that the moments and shearing forces over successive supports in unloaded segments have opposite signs, and that the moments over supports of loaded segments are always positive (as compared with the effect of the directly applied load), it follows that the first condition for a maximum is that *loaded spans should alternate with unloaded spans*, and the second, that *the loaded spans be completely covered.*

468. The following plan of dealing with the effect on the chosen unloaded span of the loads imposed on other spans naturally suggests itself:—

Number the spans off consecutively to right and left of the chosen span A′A″ of *Fig.* 208, then the following results will be intelligible after what has been already said.

Over A′ there will be a negative maximum if the odd spans to the left and the even spans to the right be fully loaded. The same distribution gives a positive maximum over A″. This is illustrated in Scheme III. of *Fig.* 208. The shearing fórce will be a negative maximum over A′ under the same conditions.

469. Applying the method described in preceding paragraphs, it will be easy to verify the results illustrated in the four schemes of *Fig.* 208, *viz.*:—

Scheme I.

With the arrangement of loads shown, *viz.*, the even spans on either side the chosen unloaded span A″A′ fully loaded, we have a negative maximum bending moment between the fixed verticals x and y.

Scheme II.

With the arrangement shown in Scheme II., we have a positive maximum shearing force over A′ and a negative over A″; a positive maximum bending moment between y and A′; a negative maximum bending moment between A″ and x.

Scheme III.

With the arrangement shown in Scheme III. we have a negative maximum shearing force over A′; a negative maximum bending moment between y and A′; a positive maximum bending moment between A″ and x.

Scheme IV.

With Scheme IV. there is a positive maximum bending moment over A′ and A″.

470. Some of the results arrived at in para. 468 are illustrated in *Fig.* 207, thus—

Scheme 1.

Scheme 1 shows a negative maximum bending moment between the fixed verticals x and y, the beam being fully loaded.

Scheme 2.

A positive shearing maximum at any chosen section C, when the left segment is fully loaded and the right unloaded, and *vice versâ*; a positive maximum bending moment at the cross section between y and A′ at which the inflexion point falls when a concentrated load is placed at C; a negative maximum bending moment for the inflexion point between A″ and x corresponding to a concentrated load placed at C.

Scheme 3.

Gives the reverse results to Scheme 2.

471. By combining Schemes I. and 1, II and 2, III. and 3, IV. and 1, total maxima may be obtained. Thus, for instance—

The shearing force is a negative maximum at any given cross section of a given intermediate span of a continuous girder when the uniform travelling load extends from this section to the right support, the left segment of the girder being unloaded, and the other spans to right and left alternately loaded, that is, the adjacent span to the right being unloaded and that to the left loaded; a positive maximum shearing force occurring at the section in question when the left segment of the given span is loaded, the right adjacent span being fully loaded and the left unloaded, and the other spans alternately loaded and unloaded (Schemes III. and 3).

Similarly, a negative maximum bending moment is obtained at any cross section of the girder between the fixed verticals x and y, when the girder is fully loaded and the other spans to right and left are alternately unloaded and loaded, the adjacent span on either side being unloaded (Schemes I. and 1).

In practice such a loading is scarcely likely to occur. If we suppose the rolling loaded divided into two portions only, which arrangement would cover the case of railway bridges with double lines of rails, the above results will read as follows :—

The shearing force at any cross section of the given intermediate span of a continuous girder will have a maximum negative value when the

load reaches from the right support up to the section in question, and the left adjacent span is fully covered; a positive maximum occurring when the load reaches from the left support to the section in question and the right adjacent span is fully loaded.

And the bending moment will have a maximum negative value at any cross section between the "fixed" verticals when the span in question is fully loaded, and the next span but one on either side is also fully loaded.

Students are referred to *Chalmers' Graphical Determination of Forces* and *Du Bois' Graphical Statics*, from which the above investigation has been taken, for further information on this subject.

472. The advantages and disadvantages of continuity in girders are discussed in para. 353 of Vol. I. The following remarks, taken from Cols. Wray and Seddons' *Instruction in Construction* may be added to them:—

"The danger of continuity lies in the fact that a slight subsidence in, or error in laying out, the levels of any of the points of support, will cause the points of contraflexure to shift, thus throwing out all the calculations and producing stresses the amounts of which cannot be foreseen.

"Any accident to the girders over one span of a bridge, formed of continuous girders, may involve the destruction of several spans, or even of the whole structure, as was the case in the Tay Bridge disaster in December, 1879, when the entire train and all the high girders over 13 spans fell into the river Tay.

"Any continuity or fixing the ends of cast-iron girders, not especially designed to meet the change of stress on either side of the points of contraflexure, should be carefully avoided, or the result might be failure of the girder; also timbers which might possibly at some future time be cut through in carrying out repairs, or otherwise, should never be allowed to depend on their continuity for strength or stiffness."

473. The question of Rivetted Joints is dealt with in Chapter XIX., Vol. I., and also in the next Chapter.

Examples.

An example of each of the cases referred to in para. 456 is added.

Ex. I. This is an example of Case I. of para. 456. . Each of the five

spans of the Continuous Girder shown in *Plate* XXXIX. is dealt with separately on the supposition that it alone is for the time being loaded and all the other four unloaded. The positive and negative moments over the several points of support are then added together, and the maximum resultant moment diagram described. (*Fig.* 212).

In this example a load of 77½ tons is placed on each span, and each span is dealt with separately in *Figs.* 210*a* to *e*, the load being placed at the points P_1, P_2, P_3, P_4 and P_5 in the respective spans A_0A_1, A_1A_2, A_2A_3, A_3A_4 and A_4A_5.

The following are the successive steps to be taken :—

(1). The "fixed" verticals are first set out, as in *Fig.* 209.

(2). A force polygon, with any convenient pole distance, is then described, as in *Fig.* 211 (in which the pole distance is taken at 100 units of length) and the moment polygons drawn, as $A_0X_1A_1$, $A_1X_2A_2$, $A_2X_3A_3$, &c. (*Fig.* 210*a* to *e*).

(3). The several deflection polygons of imposed loads and intercepts on verticals are then described by the method explained in para. 455, by setting off the length of the span on either side the point of application of the load, and the intercepts are then reduced in the manner explained in para. 459, span A_3A_4 being taken as the standard span.

(4). The maximum resultant moment diagram, *Fig.* 212, can then be drawn, the maximum resultant moment over a support being the algebraic sum of the moments over that support, due account being taken of the direction in which the moments severally act.

(5). The deflection polygon for the whole girder is then described, *Fig.* 213, by the method of para. 156, and *Fig.* 212 thereby checked.

(6). The actual resultant moment areas are then drawn in, *Fig.* 214, and the elastic polygon, *Fig.* 215, described (para. 461).

(7). The elastic curve is then interpolated, and the maximum deflections of the several spans determined by para. 119.

It is perhaps scarcely necessary to accentuate the importance of employing scales *as large as possible.*

Ex. II. *Plate* XL. shows the construction for determining the elastic curve of the cross girders of the Plate Girder Bridge, designed in Example II. of Chapter XXVII., the cross girders being supposed to be fixed at the ends and the weight of the bridge added to the applied load.

Fig. 217 shows the construction when both the lines of way are fully loaded, and the load on the girders is therefore symmetrically placed; and *Fig.* 222 shows it when the left pair of rails only are loaded, the right pair being unloaded, and the loading, therefore, unsymmetrically placed.

The following are the successive steps to be taken :—

(1). Lay out the "fixed" verticals. In the case of "fixed" beams these are each one-third the span length from either support.

(2). A force polygon with any convenient pole distance is then described, and the moment polygons A′ 1 2 3 4 5 A″A′ and A′ 1 2 3 A″A′ drawn in. The pole distance employed in *Fig.* 220, *Plate* XL. is 5 lineal units, and the moment polygons are shown in chain lines (*Figs.* 217 and 222).

(3). The several portions of this moment area, divided off by vertical straight lines drawn along the lines of loads, are then reduced to a base length of one-half the span by the method described in para. 458, half the span length being set down on the verticals through the supports. The sum of the heights of the reduced rectangles thus obtained, as PX, *Figs.* 217 and 222, represents the area of the moment polygon to the given base length (one-half the span), and may be taken as the load line of the second force polygon. It will be observed that the length of PX in this example is arrived at in a slightly different manner to that in which the load line is determined in the similar example given in the Addendum to Chapter VIII. In that case the length PX is obtained by reducing the figure of the moment polygon directly to a rectangle with base equal to one-half the span, and then dividing PX proportionally to the areas of the several parts of the polygon.

(4). The intercepts on the verticals through supports are then determined by the method described in para. 455 by setting off the span length either side of P, as PA_1' and PA_1'' and joining A_1' and A_1'' with X and producing to meet the verticals in B″ and B′ respectively. The intersection of the straight lines A″B′ and A′B″ with the fixed verticals x and y give the points x_1 and y_1 which, being joined and produced, the intercepts A″a'' and A′a' are obtained. But it will be observed that in the case of symmetrical loading, the resultant load line PX being centrally placed, we have $a'a''$ parallel to A′A″ and A″a'' = A′a' = $\frac{1}{3}$ A′B′ or $\frac{1}{3}$ A″B″.

(5). The elastic curve can now be drawn in. The load line PX (= A$'a$ of *Fig.* 219) is divided in the points s, a_1, a_2, a_3 and a_4 so that the parts sa_1, a_1a_2, a_2a_3 and a_3a_4 are proportional to the areas of the corresponding *resultant* upward and downward partial moment areas in the representative polygon a'A$'$ 1 2 3 4 5 A$''a''a'$ and the force polygon shown in *Fig.* 219 with pole distance equal to one-third the span is described. The corresponding elastic polygon is shown with a firm line.

Figs. 222 and 223 are dealt with in the same way.

N.B.—The Student's attention is called to *Fig.* 221 which shows how the irregular rectilineal figure ABCD is reduced to a triangle AaD of equal area and height, namely, by drawing through B a straight line Ba parallel to CA and then joining Aa. For the triangle BAa is equal to the triangle BCa, being on same base Ba and between same parallels Ba and CA. To each add the part DABa ; therefore the whole DAa is equal to the whole DABC.

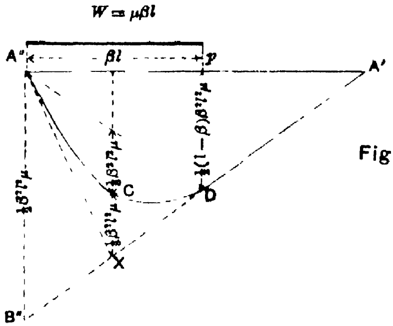


Fig 205

Fig. 206.

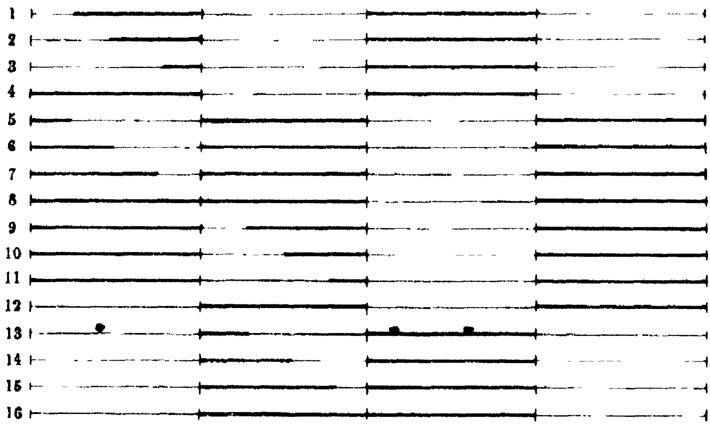


Fig. 207.

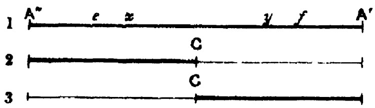
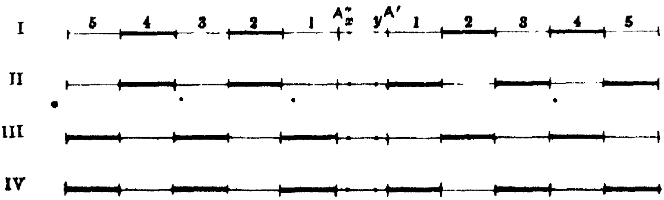


Fig. 208.



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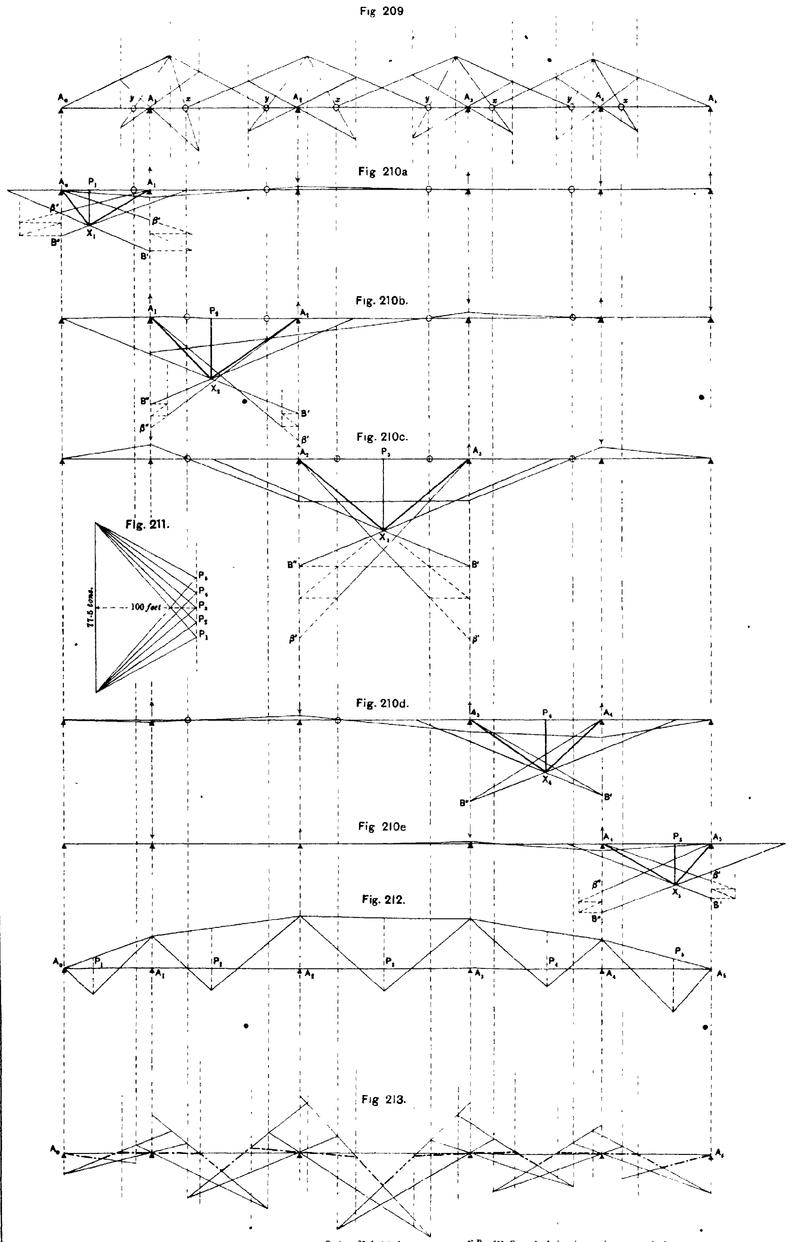

Scales—80 feet to 1 inch
40 tons to 1 inch

N.B.—(1) Span A_3 A_4 has been taken as standard span
(2) The arrows shown above the points of support indicate the direction of the moments thereat

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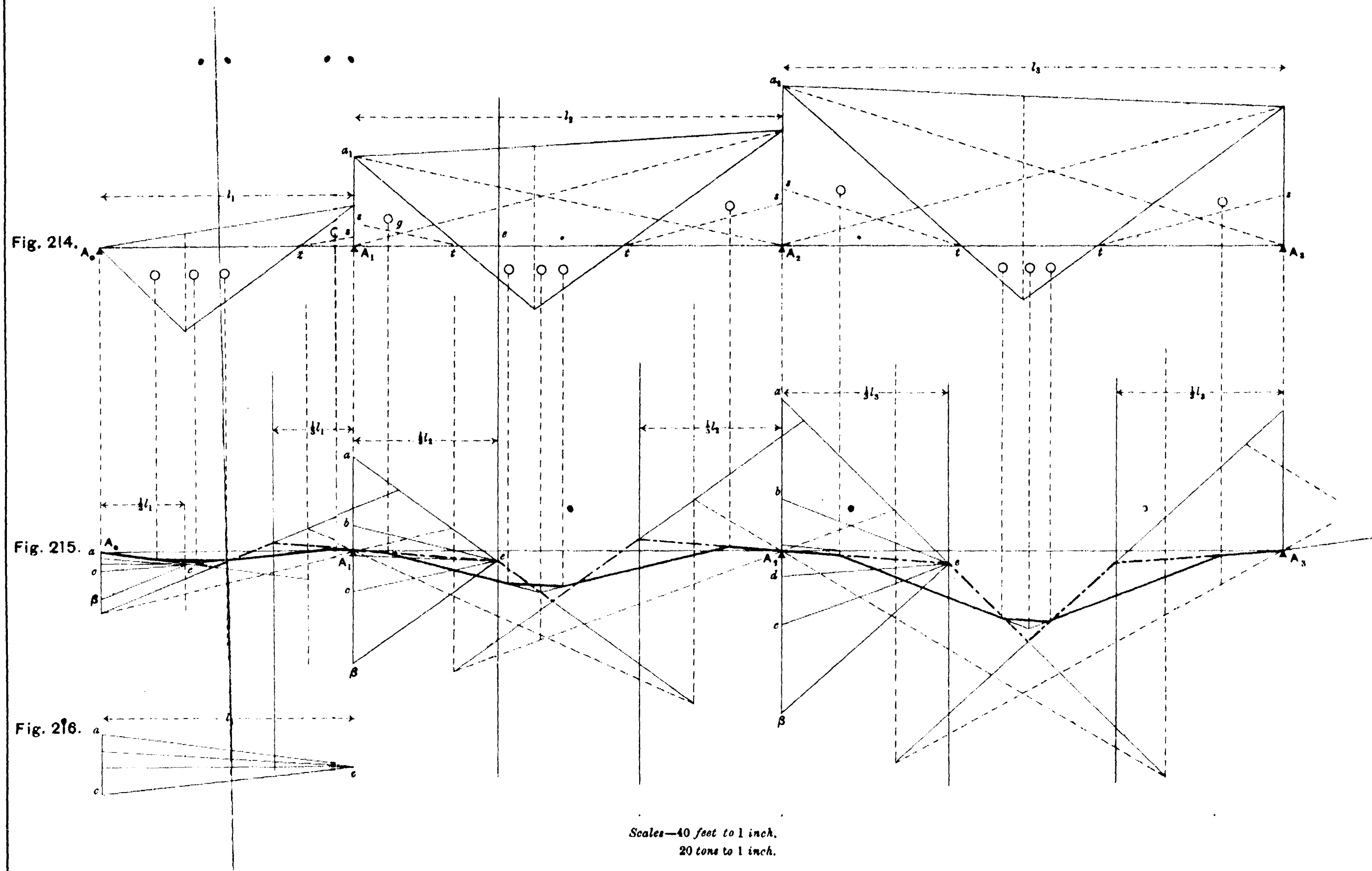
Fig. 214.

Fig. 215.

Fig. 216.



Scales—40 feet to 1 inch.
20 tons to 1 inch.

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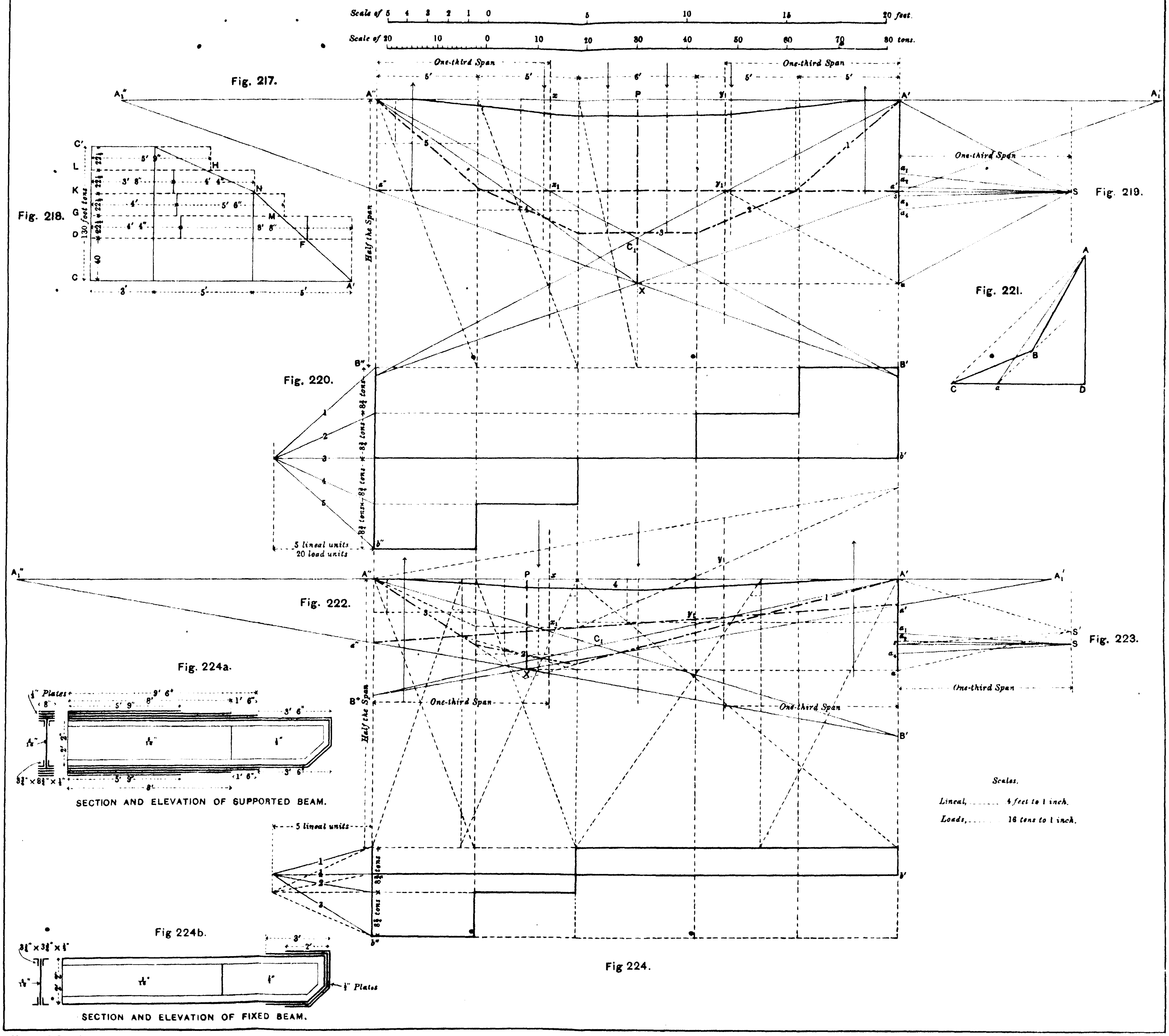
CROSS GIRDERS OF PLATE-IRON RAILWAY BRIDGE.



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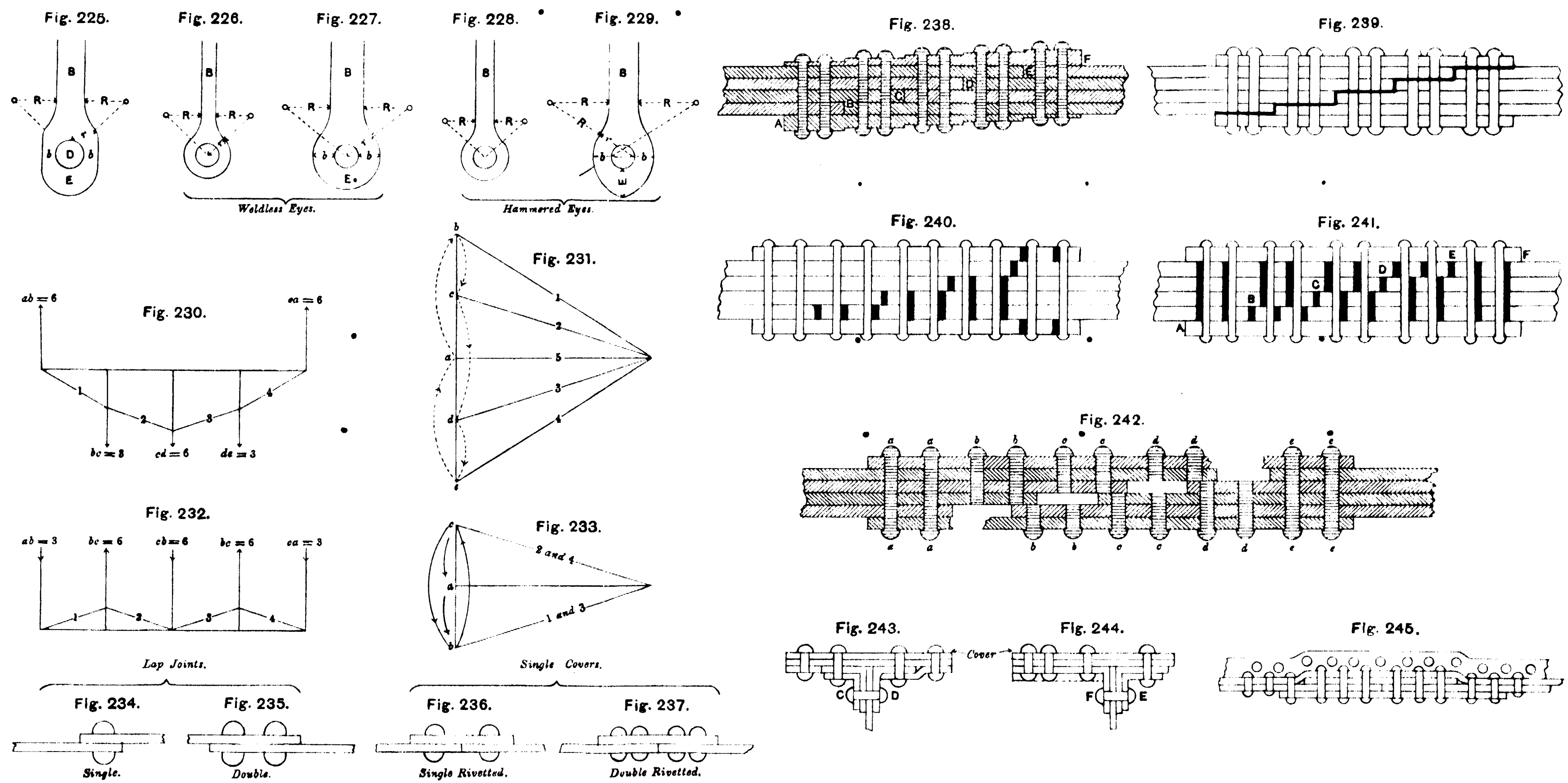
JOINTS IN IRONWORK.



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CHAPTER XXVII.

PARALLEL GIRDERS.

474. Beams of wrought-iron and steel are built up of pieces whose sections are of the various forms * described in para. 399, Vol. I., such as plate, bar, angle, tee, channel, &c., connected together by rivets or bolts of iron or steel.

They may be divided into the two principal types of (para. 266, Vol. I.)—

I. Plate Girders, or beams with continuous or solid webs of plate-iron or steel.

II. Braced Girders, or beams with open webs, of which Lattice, Warren, and N, or Whipple-Murphy, girders are instances.

These two types may be further sub-divided into girders with straight or parallel flanges, and those with a curved flange, or curved flanges, such as bowstring and saddle, or hog-backed girders, so frequently used on railways.

475. When designing built-up bridges of this nature, the following points † deserve special attention: most of them have been already referred to in Vol. I., but are collected here for convenience of reference.

Points to be looked to in the Design of Built-up Bridges.

1. Care should be taken to use, as far as possible, *the sections and lengths of iron or steel*, which are *usually to be found in the market*, bearing in mind, however, while doing so, that the use of special irons may possibly, under certain circumstances, prove more economical, by reducing the number of joints, or the weight or number of cover-plates or angle covers, required (para. 399, Vol. I.)

2. *The length of the beam*, for purposes of calculation, should be the

* *Quadrant iron* may be added to the list. It consists of a flat bar bent to the form of a quadrant with two flat flanges. Four bars are rivetted together to form a tube with four wings or feathers, in which form it is especially adapted for struts.

† Wray and Seddon's *Instruction in Construction*, 3rd Edition, pages 204 and 208 to 211

length from centre to centre of bearings, and not the clear span (paras. 275 and 396, Vol. I.)

3. The *effective depth* at any section of the girder, is the depth between the centres of pressure of the flanges, and these may, without sensible error, be considered to be at the centres of figure of the respective flange sections (paras. 186 and 204, Vol. I.)

The depth of parallel girders is usually from $\frac{1}{10}$th to $\frac{1}{15}$th the length for single span girders, and from $\frac{1}{15}$th to $\frac{1}{20}$th for continuous girders, $\frac{1}{12}$th being generally regarded as the most advantageous proportion. If the beam has a greater depth, the web is more liable to buckle, and an extra thickness is required, or stiffeners have to be rivetted to it, till, in very deep girders, the stiffeners may require nearly as much metal as the bars of an open web girder; if the beam has little depth, more metal has to be put into the flanges (para. 195, Vol. I.)

4. *Rivet holes* should be arranged so as to weaken the plates, &c., as little as possible. Drilled holes are always desirable where many plates have to be rivetted together, as they can be put closer together, and the arrangement of the joints is thereby frequently facilitated, whilst the cost is not much increased owing to the facility with which the holes can be made by the use of multiple drilling machines. In no case, however, must they be so close as not to be readily got at on both sides for the purpose of being rivetted up or that the heads of two adjoining rivets are in each other's way.

In calculating the strength of the *tension flange*, the areas of the rivet holes must be deducted, the loss due to this cause in the case of a chain-rivetted flange joint being sometimes as much as 15 per cent. of the whole.

The rivet holes in the *compression flange* need not be deducted if the work is properly done, as, being filled up by the rivets, the compression is conveyed through them from one side of the hole to the other (para. 364, Vol. I.)

5. The length of *flange plates* in compression varies from 8 to 12 feet, that of those in tension from 12 feet upwards, the object of the greater length in the latter case being to reduce the number of joints in the tension flange, in which, being a source of weakness, they should be as few as possible.

The plates in each flange must break joint, and should, as far as pos-

sible, be uniform in dimensions, and the rivets so arranged that the pattern for each plate be, as far as possible, the same. Any departure from this rule involves the risk of mistakes by the workmen. It is always considered worth while to waste a little metal in order to secure uniformity of work generally.

When the quantity of metal required in the flanges is great, it would appear better to use a large number of thin plates than a small number of thick ones, the former being stronger in proportion to their thickness than the latter, allowing of a more gradual reduction of the quantity of metal towards the points of support, and thus rendering the deflection of the beam under its load more gradual. It is not usual, however, to use any plates less than $\frac{1}{4}''$ or more than $\frac{3}{4}''$ in thickness, as very thin plates are liable to corrode, whereby their strength becomes seriously affected, whilst the bolts or rivets must be very large to give sufficient bearing area. Thin large plates are also apt to lose their flatness during the rivetting of their edges.

6. The *width of the tension flange* may, if necessary, be reduced to what is required to give a proper bearing on the piers, or, if the beam is suspended from the top flange, to a still smaller breadth.

7. The *width of the compression flange*, when not stiffened laterally, as when the cross girders are on the bottom flange, must be sufficient to prevent buckling sideways, (*i.e.*, in a horizontal plane) under the thrust. This is found from experience to be obtained by a width of $\frac{1}{30}$th to $\frac{1}{40}$th of the length of the beam. In very deep girders a smaller width may be adopted, the top flange being stiffened against lateral bending by cross girders.* The plates should have a thickness of at least $\frac{1}{700}$th the span to prevent buckling vertically, (para. 195, Vol. I.)

8. The *thickness of web plates* usually varies from $\frac{1}{4}''$ to $\frac{3}{4}''$, but it is not desirable to use plates less than $\frac{5}{16}''$ thick for open-air girders. Horizontal joints in the webs of open-air girders are objectionable, as the upper edges of the covers afford a lodgment for rain. They are necessary only in girders of unusual depth. Reducing the number of joints in the web has the advantage of reducing the number of places where rain can penetrate.

9. It is undesirable in girder work to use *angle-iron* of a thickness less than $\frac{1}{8}$th the width of one side.

* Unwin's "Iron Bridges and Roofs," p. 56.

10. *To lessen the evils of corrosion*, it is advisable to add $\frac{1}{16}''$ to thin plates and to the webs of plate girders, or $\frac{1}{8}''$ when much exposed to smoke from engines or other deteriorating influences, and all built-up work should, as far as possible, be so designed that all parts of it can be got at periodically for scraping and painting.

11. *Camber.*—Built-up beams, unless very small, are almost always constructed with a "camber" such that, when the beam is fully loaded, it becomes horizontal or nearly so. This "camber" is made equal to, or a little more than, the calculated deflection of the beam, the deflection in this case including a considerable amount of permanent "set," due to the large number of parts and the unavoidably imperfect nature of their connections. The effect of these two causes is to reduce the value (E) of the modulus of elasticity from 29,000,000 lbs. to about 18,000,000 lbs. (E = from 16,000,000 lbs. to 18,750,000 lbs.*) and in calculating the camber, this value of E should be used. With a beam of uniform strength, under stresses of 5 tons and 4 tons per square inch in tension and compression respectively, and having a depth of from $\frac{1}{10}$th to $\frac{1}{15}$th the span, the camber would be from about $\frac{1}{700}$th to $\frac{1}{400}$th the span; and if (as may be inferred from the value of E being reduced by about $\frac{1}{3}$rd) about $\frac{1}{3}$rd of this is absorbed by permanent set, the true deflection would be from about $\frac{1}{1050}$th to $\frac{1}{600}$th the span. It may be wise to increase this camber a little, say by one-third, the calculations for deflection being approximate only, as it is better to have too much than too little camber, (para. 293, Vol. I)

12. The *cross section* must be designed so as to allow of the plates being built up *seriatim*, the rivet holes being accessible on both sides.

13. *Expansion arrangements.*—As already explained, beams, if "fixed" with a view to increasing their strength and stiffness, should still be free to expand and contract lengthwise. This may be effected as explained in paras. 393, 394, Vol. I., and 351, Vol. II. If a girder were immovable at both ends, a variation of 15° Fahrenheit would produce a variation of $\frac{1}{10000}$th in length, and consequent stress of 1 ton per square inch of section.

Girders of railway bridges over 80 feet in length generally have expansion rollers under one end, but timber bed-plates are often preferred up to 150 feet, as rollers on metal plates are apt to become rigid. For

* Rankine's "Civil Engineering," 1862, p. 521.

shorter spans, timber, tarred felt, lead or planed cast-iron plates are often used under one or both ends, to lessen friction and allow of easy expansion and contraction.

Parallel Girders.

476. The subject of Parallel Girders has been so fully dealt with in Vol. I. that it only remains to add a few remarks to those contained in Chapter XIX., Vol. I., on Rivetted Joints, and illustrate with examples, also to make a few further observations on Areas and Moments of Resistance, and on Moments of Inertia.

The principles to be complied with in the design of Joints in Ironwork generally are stated in para. 367, Vol. I., but require a little further explanation and addition in their application to *Grouped Joints*, which may be defined as those in which several butt joints are grouped together under two cover plates, which are common to all of them ; there are, further, certain peculiarities in grouped joints which require attention.

477. In designing these joints, the thickness of the covers must be first determined, and the three conditions of strength enumerated in para. 367, Vol. I., fulfilled.

I—*Thickness of covers.*

From para. 366, Vol. I., it is evident that if there are two or more plates under two covers, and only one of them jointed, as in *Fig.* 57*c*, *Plate* X, Vol. I., the stress in the two covers must be inversely proportional to their respective distances from the centre of the joint, and that the total stress in the covers must equal that in the jointed plate, the intermediate unjointed plate, or plates, acting as distance piece or pieces.

On this principle the thickness of the cover of a grouped joint would, theoretically, be stepped in section, and somewhat as shown in *Fig.* 238, *Plate* XLI. This, however, would be quite inadmissible in practice, and, as explained in the para. referred to, cover plates are always made of uniform thickness, generally the same if possible as that of the plates to be joined, but *at least* as thick as the greatest thickness given by equations (23*a*) and (24) of para. 366, Vol. I.

Thus, suppose there is a pile of ν plates, each τ thick, to be joined under two covers, the upper of which is τ' and the lower τ'' thick. Let

the breadth of the plate be b. Then from equation (23a) above referred to, we have

$$\tau' + \tau'' \text{ must be rather} > \tau, \text{ and } \textit{at least}, \text{ therefore}, = \tau.$$

Suppose the top plate of the pile were the only one to be joined, the others acting as distance pieces. Then, by equation (24) above referred to, we have—taking moments about the neutral surface of the lower cover, and considering the upper cover in relation to the joint in the plate next beneath it:—

$$b\tau' \left(\tfrac{1}{2}\tau'' + \nu\tau + \tfrac{1}{2}\tau'\right) = b\tau \left\{\tfrac{1}{2}\tau'' + \left(\nu - \tfrac{1}{2}\right)\tau\right\}$$

Substituting for τ'' in terms of τ', as above, we have

$$\tau' (2\nu + 1) = 2\nu\tau - \tau'$$

$$\therefore \tau' = \left(\frac{\nu}{\nu + 1}\right)\tau \quad \text{.........................} \quad (1).$$

In other words, if the plate next beneath the top cover were the only jointed plate in the pile, that cover would have to be made at least $\left(\frac{\nu}{\nu + 1}\right)\tau$ thick.

In a similar way it may be shown that, were the plate next but one to the top cover the only one to be joined in the pile, the thickness of that cover would have to be $\left(\frac{\nu - 1}{\nu + 1}\right)\tau$, and were the plate next but two to the top cover the only jointed one, a thickness of $\left(\frac{\nu - 2}{\nu + 1}\right)\tau$ would be required for that cover, and so on. So that the several thicknesses of cover necessitated by the successive jointed plates would stand in order as follows, (reckoning away from the cover plate):—

$$\left(\frac{\nu}{\nu + 1}\right)\tau, \left(\frac{\nu - 1}{\nu + 1}\right)\tau, \left(\frac{\nu - 2}{\nu + 1}\right)\tau \ \cdots \ \left(\frac{2}{\nu + 1}\right)\tau, \left(\frac{1}{\nu + 1}\right)\tau.$$

Hence, since each cover plate is to be made of uniform section throughout, its thickness must be at least equal to $\left(\frac{\nu}{\nu + 1}\right)\tau$, and the combined thicknesses of the two cover plates $= \left(\frac{2\nu}{\nu + 1}\right)\tau$, which, for all grouped joints, is $> \tau$.

II.—*Conditions of Strength.*

Let N′ be the total number of rivets required in bearing; N″ that in shearing; and N the actual total required, being equal to the greater of N′ and N″.

Let n' be the number of rivets required in each end group, n'' that in each intermediate or central group, and m that in each transverse row.

Let ν be the total number of plates to be joined under the two covers; T′ the total stress to be met by them; and σ the stress on each plate, so that $T' = \nu\sigma$.

Let τ be the thickness of each plate; τ' that of each cover plate; and d the diameter of each rivet.

And lastly let s_t, s_c, s_b and s_s have the meanings assigned to them in Vol. I.

Then, it is evident that the total number in any case (N′ or N″)

$$= \{ 2n' + (\nu - 1)\, n'' \} \text{ .... (3).}$$

In para. 367, Vol. I., n is taken to denote the "whole number of rivets on one side of the joint," the meaning of which is at once evident in the case of a *single* jointed plate, as in *Figs* 57*b* and 57*c*, *Plate* X., Vol. I., but is, perhaps, not quite so apparent in the case of a *grouped joint*, the figure of which is stepped in section, as in *Fig.* 57*d*, *Plate* X., Vol. I., or the *Figs.* 238 to 242, *Plate* XLI.

The value of m will be known, since b is given, by the rules for pitch, for which *see* para. 369, Vol. I., and para. 481 following. The first condition can, therefore, be fulfilled. Thus:—

Condition 1°. Sufficient tensile or compressive strength in the plate is assured, provided:—

$$\left.\begin{array}{l}\dfrac{T'}{\nu}\ (= \sigma) \text{ is not} > (b - md)\, \tau s_t \text{ for tension plates, or not} \\ \qquad\qquad > b\tau s_c \text{ for compression plates, ...}\end{array}\right\} \text{...(4).}$$

Condition 2°. It is evident from *Fig.* 241, that the joint might open owing to the rivets cutting into the plates, the covers remaining intact, if the bearing surface of the plates, exposed to pressure, be insufficient. Hence, as explained in para. 367, Vol. I., the available bearing strength on one side of the stepped joint must be at least equal to the tensile strength of the plates. From *Fig.* 241 it will be seen that the available bearing surface from A to B $= \nu n'\tau d$; that from B to C $= (\nu - 1)\, n''\tau d$; that from C to D $= (\nu - 2)\, n''\tau d$, and so on; the available bearing surface at last joint being $= n''\tau d$.

Therefore, the whole available bearing strength

$$= [\nu n' + \{(\nu - 1) + (\nu - 2) + \dots 1\}\, n''] \,\tau d s_b,$$

whence, for condition 2°, in order to ensure sufficient bearing strength, we must *at least* have,

$$T' = \nu\sigma = \left\{ \nu n' + \frac{\nu}{2}(\nu - 1)\, n'' \right\} d\tau s_b = \frac{\nu}{2}\{2n' + (\nu - 1)\, n''\} d\tau s_b, \text{ (5),}$$

or, in terms of N′ we have

$$T' = \frac{\nu}{2} N' d \tau s_b = \nu \sigma, \quad \text{......................} \quad (6).$$

Now, the expression $d\tau s_b$ can be evaluated once for all and tabulated for different values of d, s_b and τ (*see* Table, para. 480); calling this expression, therefore, K′ we have for the general relation—

$$T' = \nu\sigma = \frac{\nu}{2} N'K', \quad \text{......................} \quad (7).$$

Condition 3°. It is evident that separation by shearing cannot take place along the stepped joint unless all the rivets shear simultaneously, *Fig.* 239. Thus for sufficient shearing strength we must *at least* have

$$T' = \nu\sigma = \left\{2n' + (\nu - 1) n''\right\} \frac{\pi}{4} \times d^2 \times s_s$$

$$= \{2n' + (\nu - 1) n''\} \, 0{\cdot}78 \, d^2 s_s, \quad \text{......................} \quad (8),$$

or in terms of N″ ...... $T' = \nu\sigma = N'' \times 0{\cdot}78 \, d^2 s_s = N''K''$, ...(9), in which K″ is put for the expression $0{\cdot}78 \times d^2 \times s_s$, which may be tabulated for different values of d and s_s.

Conditions 4° *and* 5°. In condition 2° we have examined the chances of failure owing to insufficiency of bearing surface, and in condition 3° that owing to want of sufficient shearing strength. It remains to examine combined chances of failure. Thus, for instance—

(*a*). The covers might tear and the intermediate groups of rivets shear off, as in *Fig.* 242.

(*b*). The rivets might cut into both covers of an end group, and at the same time into the plates on one side or other of the joint in the central groups, as illustrated in *Fig.* 240, which shows separation at the joint owing to the right group of rivets having cut into the cover plates, and the central groups into the plates on the right side of the joint.

With regard to (*a*) the tensile strength of the covers is $= 2 (b - md) \tau' s_t$, and the shearing strength of a rivet $= \frac{\pi}{4} d^2 s_s = K''$ suppose. Hence we must at least have

Condition 4°. $T' = \nu\sigma \leq 2 (b - md) \tau' s_t + n'' (\nu - 1) K''$, ...(10).

With regard to (*b*), the bearing strength of the covers for one end group only, as shown in *Fig.* 240 is $2n'\tau' d s_b = 2n' K_1'$ suppose, and that of the intermediate groups is $= \{1 + 2 + 3 + \ldots\ldots (\nu - 1)\}$ $n'' \tau d s_b = \frac{\nu}{2} (\nu - 1) n'' \tau d s_b = \frac{\nu}{2} (\nu - 1) n'' K_1'$ suppose, in which K_1' is

the tabulated value of K′ for thickness τ' and K_2' that for thickness τ.

Hence—

Condition 5°, for bearing strength of plates and covers combined we have

$$T' = \nu\sigma \leqq \left\{ 2n'\tau' + \frac{\nu}{2}(\nu - 1)\, n''\tau \right\} ds_b = 2n'K_1' + \frac{\nu}{2}(\nu - 1)\, n''K_2', \quad (11).$$

478. The conditions of strength above discussed are here collected for convenience of reference.

Strength of covers. The thickness of each of the two covers required to cover the "grouped" joint of a pile of ν plates, each τ thick, must not fall short of $\left(\frac{\nu}{\nu + 1}\right)\tau$.

Strength of Plates.

1°, (tensile or compressive strength of plates)

$$\frac{T'}{\nu} = \sigma \leqq \left\{ \begin{array}{c} (b - md)\,\tau s_t \text{ in tension} \\ b\tau s_c \text{ in compression} \end{array} \right\}$$

2°, (bearing strength of plates)

$$T' = \nu\sigma \leqq \frac{\nu}{2} N'd\tau s_b \left(\text{or } \frac{\nu}{2} N'K'\right).$$

3°, (shearing strength)

$$T' = \nu\sigma \leqq N'' \times 0{\cdot}78\, d^2 s_s \text{ (or } N''K'').$$

4°, (1° for covers and 3° for plates)

$$T' = \nu\sigma \leqq 2\,(b - md)\,\tau' s_t + n''\,(\nu - 1)\;0{\cdot}78 d^2 s_s.$$

5°, (2° for covers and plates)

$$T' = \nu\sigma \leqq \left\{ 2n'\tau' + \frac{\nu}{2}(\nu - 1)\, n''\tau \right\} ds_b.$$

As a matter of fact, the total number of rivets N is practically found from 3°, and their distribution from 4°, the other equations giving results which are less than those given by 3° and 4°.

479. In the flanges of large girders, frequent "grouped" joints must necessarily occur, and we shall now examine how, having made the calculations above described for the joint comprising the largest number of plates, those comprising fewer plates may be dealt with in the most expeditious manner.

The stress in each plate is by hypothesis the same throughout the flange, and equal to the total stress to be met divided by ν, the number of plates. It follows, then, that N′ is constant throughout the pile, since it varies as σ, and that N″ varies directly with the number of plates, this being evident from the relation sestablished in the preceding paragraph, (2° and 3° above).

But, if n'' be the number of rivets in an intermediate group of ν plates, then the number n_1'' in a group of ν_1 plates is given by the following relations :—

Condition 4° for ν plates gives

$$\nu\sigma \leqq 2\ (b - md)\ r' s_t + n''\ (\nu - 1)\ dr s_b.$$

Assuming r' to be the same throughout the flange, the same condition for ν_1 plates gives

$$\nu_1 \sigma \leqq 2\ (b - md)\ r' s_t + n_1''\ (\nu_1 - 1)\ dr s_b.$$

$\therefore$ subtracting we have $(\nu - \nu_1)\ \sigma \leqq \{n''\ (\nu - 1) - n_1''\ (\nu_1 - 1)\}\ dr s_b$.

Hence $$n_1'' = n''\left(\frac{\nu - 1}{\nu_1 - 1}\right) - \left(\frac{\nu - \nu_1}{\nu_1 - 1}\right)\frac{\sigma}{dr s_b}, \quad \text{..............} (12).$$

Also $N_1 = \frac{\nu_1}{\nu} N$ and, further, $N_1 = 2n_1' + (\nu_1 - 1)\ n_1''$. Hence N_1, n_1' and n_1'' are known.

480. The following Table,* prepared by Captain H. R. Sankey, (late R.E.), gives the values of K' and K'', referred to in the previous paragraphs, *worked out for the particular values* $s_b = 5$ *tons, and* $s_s = 4$ *and* 5 *tons per square inch.* These results may, of course, be reduced to suit any other values of s_b and s_s.

Table giving bearing strength of Plates (K′) *and shearing strength of Rivets* (K″).

Diameter of rivet.	Resistance in tons to single shear. $K'' = 0{\cdot}78\ d^2 s_s$		Bearing Resistance in tons at 5 tons per square inch of a single rivet hole ($K' = dr s_b$) for plates from $\frac{1}{16}''$ to 1″ thick.								
	Wrought-iron, s_s = 4 tons per sq. inch	Steel, s_s = 5 tons per sq. inch.	$\frac{1}{16}''$	$\frac{1}{8}''$	$\frac{1}{4}''$	$\frac{3}{8}''$	$\frac{1}{2}''$	$\frac{5}{8}''$	$\frac{3}{4}''$	$\frac{7}{8}''$	1″
$\frac{1}{2}''$	0·785	0·98	0·156	0·312	0·625	0·937	1·250	1·562	1·875	2·187	2·500
$\frac{5}{8}''$	1·227	1·534	0·195	0·391	0·781	1·172	1·562	1·950	2·344	2·734	3·125
$\frac{3}{4}''$	1·767	2·208	0·234	0·469	0·937	1·406	1·875	2·344	2·812	3·281	3·750
$\frac{7}{8}''$	2·405	8·006	0·273	0·547	1·094	1·640	2·187	2·734	3·281	3·828	4·375
1″	8·141	3·927	0·312	0·625	1·250	1·875	2·500	3·125	8·750	4·375	5·000
$1\frac{1}{8}''$	3·976	4·970	0·351	0·703	1·406	2·109	2·812	3·576	4·219	4·922	5·625
$1\frac{1}{4}''$	4·908	6·135	0·391	0·781	1·562	2·344	3·125	3·906	4·687	5·460	6·250

In single cover joint, or in considering the outer joint only of a grouped joint, the shearing resistance of each rivet section should, for equality of strength, be equal to the bearing resistance of the corresponding rivet hole of the nearest cover (or in either plate in a lap joint); therefore, in using the above Table, which gives the relative resistance

* Wray and Seddons' "Instruction in Construction," 3rd Edition, p. 192.

of rivets to shearing and of plates to cutting *under the ordinary limits of stress*, it will generally be sufficient to calculate the number of rivets required either for shearing or for bearing, taking the one which offers the least resistance. Thus, with $\frac{3}{4}''$ rivets and $\frac{3}{8}''$ cover, the Table shows that more rivets would be required for bearing than for shearing; hence, the number required for bearing will be more than enough for shearing, whilst the reverse would be the case with a $\frac{3}{4}''$ cover.

The method of using the Table will be apparent from the following example*:—

The tension bar of a lattice girder, $5'' \times \frac{1}{2}''$ in section, is rivetted to the flanges, by $\frac{7}{8}''$ rivets and is subjected to a stress of $10\frac{5}{16}$ tons. From above Table we have, the bearing resistance of $\frac{7}{8}''$ rivets in $\frac{1}{2}''$ plate is 2·187 tons per square inch, and the shearing resistance is 2·405 tons. Hence, the least number of rivets necessary must be $10\frac{5}{16} \div 2{\cdot}187 = 4{\cdot}7$, or 5 rivets.

481. *Size and Pitch of Rivets.*—The rules given in para. 369, Vol. I, for transverse pitch are not quite satisfactory. It would appear that, in order to ensure equality of strength in all parts of the joint, the shearing strength of the rivets should equal the tensile strength of the net area of the cross section of the plate to be joined, and also, the bearing strength of one rivet hole in single, or of two in double, rivetting should likewise equal the tensile strength of the plate left between any two rivet holes in the same transverse row.

If l' = transverse pitch, and τ, d, s_b, s_s and s_t have the same meanings as before, (l' being measured from centre to centre of rivet), we have—

Tensile strength of plate between two rivet holes in the same transverse row $= (l' - d)\,\tau s_t$.

Bearing strength of one rivet hole $= d\tau s_b$.

Hence, in the case of *single rivetting*, $l' = \left(1 + \frac{s_b}{s_t}\right) d$

and, in the case of *double rivetting*, $l' = \left(1 + 2\,\frac{s_b}{s_t}\right) d$ ....(1).

Whence, since $s_b = s_t$ we have, $l' = 2d$ for single, and $3d$ for double rivetting, for *bearing and tensile strength only*.

The shearing strength of one rivet section $= \frac{\pi}{4}\,d^2\,s_s$,

Hence, for a single-rivetted single shear joint $\frac{\pi}{4}\,d^2\,s_s = (l' - d)\,\tau s_t$.

* Wray and Seddons' "Instruction in Construction," 3rd Edition, p 182.

In the case of single-rivetted double shear joints, there are two cross sections of rivet to resist the pull, and so likewise in that of double-rivetted single shear joints; in the case of double-rivetted double shear joints there are four cross sections of rivet. Therefore, if k be a constant which equals 1, 2, or 4 as the case may be, we have the general expression

$$0{\cdot}7854\, kd^2 s_s = (l' - d)\, \tau s_t$$

whence $$l' = \frac{0{\cdot}7854\, kd^2 s_s}{\tau s_t} + d, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again, if k' be a constant which, in the case of single rivetting equals 1, and of double rivetting 2, we have for equation (1) the general expression

$$l' = \left(1 + k'\frac{s_b}{s_t}\right) d, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence, equating (2) and (3), we obtain a general expression for d in terms of known quantities; thus, when the tensile strength of the net area of the cross section of the plate to be joined is equal to the shearing strength of the rivets as well as to their bearing strength,—when, in fact, the several parts of the joint are all equally strong—we have for the diameter of the rivet

$$d = \frac{\tau}{0{\cdot}7854} \times \frac{k'}{k} \times \frac{s_b}{s_s}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and putting $s_b = 5$ and $s_s = 4$ the usual values, we have $d = 1{\cdot}59 \times \frac{k'}{k} \times \tau$.

For single-rivetted single shear joints, $k' = 1$ and $k = 1 \therefore d = 1{\cdot}6\,\tau$.

For single-rivetted double shear joints, $k' = 1$ and $k = 2 \therefore d = 0{\cdot}8\,\tau$.

For double-rivetted single shear joints, $k' = 2$ and $k = 2 \therefore d = 1{\cdot}6\,\tau$, and for double-rivetted double shear joints, $k' = 2$ and $k = 4 \therefore d = 0{\cdot}8\,\tau$.

Hence, generally, in the case of single shear joints $d = 1{\cdot}6\,\tau$, and in that of double shear, $d = 0{\cdot}8\,\tau$. On account, however, of the risk of damaging the plates in punching holes of small diameter, also of the additional labour involved, and bearing in mind the necessity of maintaining as much uniformity as possible in order to avoid errors in practical construction, the same diameter (*viz.*, $1{\cdot}6\,\tau$) is generally given to the rivets whether in single or double shear.

Substituting $d = 1{\cdot}6\tau$ in (2) we have

$$l' = \frac{0{\cdot}7854\, k\,(1{\cdot}6\tau)^2}{\tau} \times \frac{4}{5} + 1{\cdot}6\,\tau \;\therefore\; l' = (k+1)\,1{\cdot}6\,\tau.$$

Thus, for single-rivetted single shear joints $l' = 3{\cdot}2\tau$, and for single-

rivetted double shear and double-rivetted single shear joints $l' = 4 \cdot 8r$; for double-rivetted double shear joints $l' = 8r$.

And substituting $d = 1 \cdot 6r$ in (1) we have $l' = 3 \cdot 2r$ for single and $4 \cdot 8r$ for double-rivetted joints—a result practically the same as for equation (2).

The above dimensions indicated by theory are slightly altered in practice, the rule generally followed being that of Mr. Fairbairn, to make $d = 2r$ for plates under $\frac{1}{2}''$ thick and $= 1\frac{1}{2}r$ if $\frac{1}{2}''$ thick or more. Professor Unwin's rule is $d = 1 \cdot 2 \sqrt{r}$ (*Machine Design*, 1877). The minimum pitch of rivets is $2d$, but in the case of punched holes is made $= 2\frac{1}{2}$ to $3d$. In zigzag rivetting the distance between the transverse pitch lines is made at least $= \frac{2}{3}$ what the transverse pitch would be for chain rivetting. In a pile of plates of different thicknesses, r in above would be the thickness of the thickest plate.

482. *Lozenge Rivetting.*—The arrangement of zigzag rivetting known as *Lozenge Rivetting* is referred to in para. 367, Vol. I., but the explanation therein given (owing to misprints) is not satisfactory. The following explanation, due to Professor Unwin, is offered to the Student.

From *Fig.* 57*e*, *Plate* X., Vol. I., it is evident that the section at s (or c) is reduced in tensile strength by one rivet hole, this strength being (with the same symbols as in para. 477, p. 272) measured by $(b - d)\, rs_t$. It is likewise evident that the second section rt (or pq) is similarly weakened by *two* rivet holes. But it is equally clear that the plate cannot fail at the second section rt (or pq) without also simultaneously shearing across the first rivet section at s (or c). Consequently the strength of the plate at rt (or pq) is not merely measured by $(b - 2d)\, rs_t$, but by $\left\{ (b - 2d)\, rs_t + \frac{\pi}{4} d^2 s_s \right\}$. If now the diameter of rivet and thickness of plate be so proportioned that $\frac{\pi}{4} d^2 s_s = d r s_t$, that is, so that the shearing resistance of one rivet section be not less than the tensile resistance of a strip of the plate one diameter wide, then will the strength of the plate at section rt (or pq) $= (b - d)\, rs_t$, that is, will be the same as at section s (or c).

In a similar way, the plate cannot fail at the third section cd (or ab) without simultaneously shearing across the two rivets at the second section rt (or pq) as well as the single rivet at the first section s (or c), so that the strength at section cd (or ab) $= \left\{ (b - 4d)\, rs_t + 3\, \frac{\pi}{4} d^2 s_s \right\}$

which, under the conditions stated above, reduces to $(b - d) \tau s_t$, that is, to exactly the same strength as that of the first cross section at s (or c).

Similarly the fourth section might have 8, the fifth 16, and the n^{th} section 2^{n-1} rivet holes, without the strength of the bar being reduced below that of the first section at s (or c), or virtually below that due to reduction by one rivet hole.

But if $\frac{\pi}{4} d^2 s_s = d\tau s_t$ at least, then $d = 1{\cdot}27 \frac{s_t}{s_s} \tau$, at least; and taking $s_t = 5$ tons $= s_s$ this reduces to $d = 1{\cdot}27\, \tau$, and if $s_t = 4{\cdot}2$ and $s_s = 3$, an extreme case, then $d = 1{\cdot}8\, \tau$, at least.

But as the value of d is in practice taken $= 1\frac{1}{2}\, \tau$ to $2\, \tau$ (*see* preceding paragraph), it is evident that by this lozenge-shaped arrangement of rivets the effective section of the plate is virtually reduced in strength by one rivet hole only, so that the n^{th} cross section may have up to 2^{n-1} rivet holes cut in it, provided the rules for pitch admit of that number.

483. The principles to be observed in the design of Rivetted-Braced Joints are explained in para. 373, Vol. I. The only point that need be mentioned here is that "the line of fracture" which, on p. 15 of Mr. Latham's *Wrought-Iron Bridges*, is defined as "the line drawn through the plate or bar (on its surface) crossing all the lines of strain in such a way that the section of the plate along it is a minimum" should be as long as possible. For "it is found by experiments on ordinary fibrous plates, that the plate is more likely to break in such a line, even though it be irregular and zigzag than in any other."

Practical Hints for Rivetted Joints.

484. The following practical hints for rivetted joints are collected here for convenience of reference, *viz.* :—

1. *Cover plates.*—If a pile of ν plates, each τ inches thick, are to be joined under two covers, the thickness of each cover plate must at least $= \left(\frac{\nu}{\nu + 1}\right) \tau$ inches, and is generally made $= \tau$ (para. 477).

2 Some joints require special devices in order to admit of the use of double covers. *Figs.* 243 and 244, *Plate* XLI., show different ways of arranging double covers to the flange plate of a girder, one cover in each case being placed on the outside of the flange, whilst the inside

of the joint may be covered as most convenient. At C, *Fig.* 243, there is room for a small *cover slip* in addition to an inner angle-iron called an *angle cover* or *wrapper*; at D a cover is cranked round the angle-iron; at E, *Fig.* 244, an angle cover alone is used; whilst at F another cover slip could be added, as shown by dotted lines, which might be made strong enough to do the work required either for a joint in the angle-iron, or in the flange plate. If an angle-iron is cranked over a cover, as shown in elevation in *Fig.* 245, the outer cover should be prolonged one or two rivet holes at each end, to compensate for the strength lost by the bend in the angle-iron.

3 Rivets should be arranged so as to admit of being got at on both sides for the purpose of clenching the rivets.

4. *Size of rivets.*—If τ be thickness of thickest plate to be joined, then Fairbairn's rule is, diameter of rivet (d) $= 2\tau$ for plates under $\frac{1}{2}''$ and $1\frac{1}{2}\tau$ for $\frac{1}{2}''$ plates or over; and Unwin's rule (*Machine Design*, 1877) is $d = 1{\cdot}2\sqrt{\tau}$.

With rivets so proportioned, the total number in single shear joints (*viz.*, lap, single cover and grouped joints) will, when the resistance to shearing is taken at 5 tons, depend on the number required for bearing with plates under $\frac{1}{2}''$ thick, and for shearing with plates $\frac{1}{2}''$ thick and over.

If $d = 1\frac{1}{2}\ \tau$, and $s_t = s_b = 5$ tons and $s_s = 4\frac{1}{4}$ tons, then will single shear joints be equally strong as regards both tearing and bearing.

Rivet holes are seldom over $1\frac{1}{8}''$ diameter even with $1''$ plates.

5. *Rivet heads* should have a diameter of from $1\frac{1}{2}d$ (where $d =$ diameter of rivet hole) for countersunk to $2d$ for conical heads, with a height of about $\frac{3}{5}d$ for snap to $\frac{7}{8}d$ for conical heads. For the method of forming ellipsoidal and segmental heads, see *Molesworth's Pocket-Book.*

6. *Rivet iron and weight of Rivets.*—Rivet iron increases in diameter by $\frac{1}{16}''$ from $\frac{1}{2}''$ to $1\frac{1}{4}''$. The diameter of the rivets should be about $\frac{1}{16}''$ less than that of the corresponding rivet holes.

The length of bar required for each rivet $=$ length of shank $+$ ($2d$ to $2\frac{1}{2}d$) for a pan head, ($1\frac{3}{16}d$ to $1\frac{1}{4}d$) for a snap or conical head, and ($1d$) for a countersunk head. The extra length required for rivet head being known, the weight of iron can be taken from any Table of Weights of round iron, or by taking it out at 0·27 lb. per cubic inch.

7. *Pitch of Rivets.*—Minimum pitch $= 2d$, but ($2\frac{1}{2}$ to 3) d is adopted for punched holes; maximum longitudinal pitch in compression plates

= 12 times the thickness of weakest outside plate. In zigzag rivetting the distance between transverse lines of rivets = at least ⅔rds the transverse pitch for chain rivetting.

8. The *distance of the outer rivets from the edge* of a plate depends on the direction of the fibre of the metal. In the direction of the fibre, the distance from the edge to the centre of the rivet hole should be not less than $\frac{1}{2}'' + r + \frac{1}{8}d$; in a direction at right angles to the fibre, the distance from edge to centre of rivet hole may be $\frac{1}{8}''$ less than this.

A safe rule is to make the end distance measured from circumference of hole = 2 diameters of rivet, and side distance $\frac{1}{4}''$ less.

Areas and Moments of Resistance.

485. *Areas of resistance.*—In para. 198, Vol. I., it is shown that the figure representing the area of longitudinal stress variation across a slightly bent beam of isotropic material, the cross section of which is rectangular, consists of two triangles, having a common apex at the point in which the neutral axis traversing the beam longitudinally meets the cross section, and may be described by drawing the two diagonals of the rectangle which will thus enclose the area of stress. It is further shown that this area may be regarded as the equivalent area of resistance of a uniformly varying stress acting over the cross section in question. The principles on which such equivalent areas may be determined for cross sections whose figures are other than rectangular will be apparent from the following :—

Draw the figure of the cross section of the beam to a convenient scale, as in the figures of *Plate* XLII., and determine the position of the centre of gravity either by cutting out the section in cardboard, as explained in para. 202, Vol. I., and suspending it in two positions, or by any other of the methods already explained.

Let NA, *Plate* XLII.,* represent an axis passing through the centre of gravity of the section. On each side of NA project successive layers of the section, taken at convenient intervals apart, on to a line drawn parallel to NA, and at a distance from it equal to that of the extreme fibre of the section. Let *cc* represent such a projection of any layer *ee*; then, since the intensity of the stress increases uniformly from zero at NA to a maximum at the extreme fibre, by joining the points *cc* with

* *Plate* XLII. is taken from Wray and Seddons' "Instruction in Construction," with slight alterations, as also the description of the method

the centre of gravity of the section, and thereby cutting the layer *ee* in the points *bb*, we obtain a length *bb* which in each case represents the total stress on the fibres *ee*. By dealing thus with successive layers of any section, taken at convenient intervals apart, the shaded areas of uniform intensity of stress, shown on the sections of *Plate* XLII., are obtained.

The numerical values of these areas may be severally ascertained by carefully cutting out the figures of the respective areas in heavy card-board, weighing them, and dividing the weight of each so obtained by that of a square inch of the cardboard.

486. *Moments of resistance.*—As explained in Chapter IX., Vol. I., the moment of resistance of the cross section is equal to the product of its area of resistance into the safe intensity of stress multiplied by the distance between the centres of stress of the cross section, as to which *see* para. 203, Vol. I.

Distribution of the Shearing Stress.

487. From the previous investigation it would appear that the equivalent area of resistance of a uniformly varying stress over a cross section of almost any figure, may be determined, and from this area the distribution of the shearing stress may be determined as follows, *viz.* :—

Let *Fig.* 246 represent the side elevation of a portion of a beam of rectangular cross section. Then, since the horizontal shearing stress along any layer, such as ee_1, is equal to the sum of the differences of the direct stresses on the fibres in the portions *fe*, ge_1 of the cross sections *ff*, *gg*, it follows that any part of the sectional area might be taken to represent the direct stress upon that area, or the difference between the stresses at two adjacent sections, so that the horizontal stress at any layer ee_1, *Fig.* 246, one unit in length, and so also the vertical shearing stress per unit of height at e_1 might be represented by an ordinate equal in length (on any convenient scale) to the number of units contained in the area of the section between *ee*, *Fig.* 247, and the outer edge *cc* of the beam.

Thus, if the shaded portion of the cross section, *Fig.* 247, represent the equivalent area of uniform intensity of stress, a diagram showing the distribution of the shearing stress throughout the section might be

drawn, as in *Fig.* 248, by laying off from the straight line dd, which represents the depth of the beam, ordinates proportional in length to the areas of the shaded portion of the figure taken in succession from the outer edges to the neutral axis of the section, as the ordinates in *Fig.* 248, which are proportional in length to the successive areas 1, 1 + 2, 1 + 2 + 3, and 1 + 2 + 3 + 4, of the shaded figure; so also for the shaded figures of the other cross sections; and by joining the ends of these ordinates the diagrams indicated by continuous lines are obtained, giving the distribution of the shearing stress throughout the section. The boundary curves will obviously approximate more nearly to the true curve the nearer together the sections are taken, that is, the more numerous the ordinates are taken, and, moreover, the area of the diagram will obviously represent the total magnitude of the shearing stress at the cross section of the beam in question.

It is evident from the above that, in a rectangular cross section, the distribution of the shearing stress may be represented by a parabola (as stated in para. 213, Vol. I.), since the ordinates of the diagram vary as the areas of similar triangles. *See Fig.* 248.

488. *Intensity of shearing stress.*—In *Figs.* 248, 250, &c., the corresponding *intensity* of shearing stress throughout the section is indicated by a dotted curve, the ordinates of which are found by dividing the ordinates of the curve representing the distribution of the shearing stress at each layer by the actual breadth of that layer at the point in the cross section to which each ordinate refers.

Since in *Fig.* 247 the breadth is uniform throughout the cross section, the curve representing the intensity of the shearing stress is merely a flatter parabola than that representing the distribution of that stress.

Moments of Inertia.

489. As the figures of the cross sections of girders of ordinary construction are usually symmetrical and simple, and may generally be supposed to be made up of rectangles, the value I of their Moments of Inertia about an axis passing through the centre of gravity of the section may often be readily determined by the following simple method, depending on a knowledge of the relation $I = \frac{1}{12}bd^3$, being the value of the Moments of Inertia of a rectangle, of breadth b and depth d, about an axis passing through its centre of gravity.

Thus, the area of the cross section of the girder shown in *Fig.* 272, *Plate* XLIV., is made up of rectangles, as follows, *viz.*:—

Area = ABCD − 2 (EFGH + KLMN + PQRS).

Therefore, the Moment of Inertia I of the area about an axis passing through its centre of gravity is equal to the algebraic sum of the Moments of Inertia of its component parts about the same axis, so that

$I = \frac{1}{12} [24 \times (66)^3 - 2 \{7{\cdot}125 \times (61)^3 + 3{\cdot}875 \times (59{\cdot}75)^3 + 0{\cdot}625 \times (50{\cdot}75)^3\}] = 574{,}992 - 421{,}808 = 153{,}184$ inch-units of inertia.

490. We shall now add a few practical examples illustrative of the principles that have been discussed in the Chapters of Vol. I. and in the preceding paragraphs. Commencing with a simple example of a Grouped Rivetted Joint and then proceeding to the complete design of a Plate Girder with parallel flanges, suitable for a Railway Bridge, we shall conclude with an example of a Box Girder. In these designs the analytical methods of Vol. I. will be employed equally with the graphical ones described in this Volume, that method being employed which would appear to be the more suitable for the immediate object in view.

EXAMPLE I.

GROUPED RIVETTED JOINT.

The tension flange of a Wrought-Iron Parallel Flanged Plate Girder is 19 inches wide and made up of four $\frac{5}{8}''$ plates, rivetted to the web by two $4\frac{1}{2}'' \times 4\frac{1}{2}'' \times \frac{5}{8}''$ angle-irons. The four plates are to be joined together under two covers, the flange being liable to a maximum bending moment of 1,188 foot-tons. The depth of the girder is 5 feet 6 inches. Design the joint so that the plates shall be weakened as little as possible, taking $s_t = 5$, $s_b = 5$ and $s_s = 4$ tons. The Table given in para. 479 may be employed.

This joint is similar in many respects to the main joint in the main girder of the Plate Iron Railway Bridge designed in Example II., following.

(1). For the *thickness of cover plates*, we have, by para. 477, $\tau' = \tau''$ = (*at least*) $\frac{4}{5}'' \times \frac{5}{8}'' = \frac{1}{2}''$. It will, however, be more convenient to make covers and plates all of same thickness, *viz.*, $\frac{5}{8}''$.

(2). For *size of rivets* we have (para. 484), $d = 1\frac{1}{2}'' \times \frac{5}{8}'' = \frac{15}{16}''$, or say 1″.

(3). For *pitch of rivets*, $l' = (2\frac{1}{2}$ to $3)\ d = 2\frac{1}{2}''$ to $3''$, say $3''$.

(4). Therefore $m = 5$.

(5). Available section of two angle irons, one on each side the web, after deducting two rivet holes in each, will be

$$2\ (2 \times 4\tfrac{1}{2}'' - \tfrac{5}{8}'' - 2 \times 1'') = 2 \times 6\tfrac{3}{8}'' \times \tfrac{5}{8}'' = 7{\cdot}97 \text{ square inches.}$$

$\therefore$ available strength of angle-irons $= 7{\cdot}97$ square inches $\times$ 5 tons $= 39{\cdot}85$, or say 40 tons.

The longitudinal stress on boom $= \dfrac{\text{bending moment}}{\text{effective depth}} = \dfrac{1188 \text{ foot-tons}}{5{\cdot}5 \text{ feet}} =$ 216 tons.

$\therefore$ stress to be taken by the four plates $= T' = (216 - 40) = 176$ tons, and stress to be taken by each plate $= \sigma = \dfrac{176}{4} = 44$ tons.

(6). If lozenge rivetting be adopted, each plate will be weakened by one rivet hole only, so that available strength of each plate will be

$$(19'' - 1'') \times \tfrac{5}{8}'' \times 5 \text{ tons} = 56{\cdot}25 \text{ tons,}$$

which is more than sufficient.

Therefore Condition 1^c is satisfied (paras. 477 and 478).

(7). For *Condition* 2° we have for N', the total number of rivets required in *bearing* (para. 478),

$$N' = \frac{2\sigma}{K'} = \frac{88}{3{\cdot}125} = 28{\cdot}1 \textit{ at least}, \text{ say 30, the nearest multiple of } m.$$

(8). For *Condition* 3° we have for N'', the total number of rivets required in *shearing*,

$$N'' = \frac{T''}{K''} = \frac{176}{3{\cdot}14} = 56{\cdot}37 \textit{ at least}, \text{ say 60.}$$

(9). Hence $N = 60$ rivets.

(10). For *Condition* 4° we have

$$2\ (19'' - 5 \times 1'') \times \tfrac{5}{8}'' \times 5 \text{ tons} + n'' \times 3 \times 3{\cdot}14 = 176 \text{ tons,}$$

$\therefore n'' = 9{\cdot}4$ rivets *at least*, or say 10, being nearest multiple of m.

(11). Hence, since $N = 2n' + (\nu - 1)\ n''$ we have

$$n' = \tfrac{1}{2}\ (60 - 3 \times 10) = 15 \text{ rivets.}$$

(12). Substituting $n' = 15$ and $n'' = 10$ in equation for *Condition* 5°, we have

$$\left\{ 2n' + \frac{\nu}{2}\ (\nu - 1)\ n'' \right\} r d s_b = \{2 \times 15 + 2 \times 3 \times 10\} \times 3{\cdot}13 = 281{\cdot}25$$

tons, which is stronger than is required, T' being $= 176$ tons only.

The joint may, therefore, be designed as indicated above.

EXAMPLE II.*

The Design of a Plate Iron Girder Railway Bridge.

Specification.

A double line of narrow gauge railway is to be carried over a clear opening of 62 feet, the effective span, or span from centre to centre of bearings, being 66 feet, and the total length of the bridge 70 feet.

The bridge is to be of the form shown in *Plate* XLIII. The main girders are to be placed on lead on the piers, and are not to be fastened in any way.

The rails are to be carried on longitudinal timbers, 12″ × 6″, which are not to be taken into account as affording any strength to the bridge; these timbers rest on longitudinal iron beams, carried on, and rivetted to, cross beams at central intervals of 5 feet 6 inches.

The cross beams rest on the lower flanges, and are rivetted to the web and stiffeners, of the main girders.

The platform between the longitudinals, cross beams, and main girders, is to consist of $\frac{1}{4}''$ plates, on which ballast is to be laid to an average depth of 4 inches.

The rolling load on the main girders and longitudinals of the bridge may be taken at $1\frac{1}{2}$ tons per foot run for each line of way.

The working resistances given in Table, p. 435, para. 397, Vol. I., are to be adopted.

Remarks.

The width of the bridge is determined by Board of Trade Rules, which are as follows:—

"Rule 15. No standing work should be nearer to the side of the widest carriage in use on the line than 2 feet 4 inches, at any point between the level of 2 feet 6 inches above the rails and the level of the upper parts of the highest carriage doors. This applies to all arches, abutments, piers, supports, girders, tunnels, bridges, roofs, walls, posts, tanks, signals, fences, and other works, and to all projections at the side of a railway constructed to any gauge."

"Rule 16. The intervals between adjacent lines of rails, or between lines of rails and sidings, should not be less than 6 feet."

* This example is taken in part from Wray and Seddons' "Instruction in Construction."

The condition, that the longitudinal sleepers carrying the rails are not to be taken into account as affording any strength to the bridge, is necessary from the consideration that these timbers may, and probably will, be renewed by men knowing nothing of their functions as part of the bridge, and consequently, if relied on for strength, their repair might involve the safety of the bridge.*

The distance apart of the cross beams has been fixed on as being likely to render the platform stiff. It would be cheaper to put them further apart, but the longitudinals would then have to be stronger. This distance or any distance up to 7 or 8 feet, has the advantage of ensuring that only one driving wheel of an engine can come on to them at one time, (para. 339).

The curved form adopted for the plates of the platform is for the purpose of drainage, and a row of small holes should be punched at the bottom of the curve, and provision made for carrying off the water.

The Longitudinal Beams.

These girders have a span of 5·5 feet; they are rivetted at the ends to the cross beams, are in the condition of "imperfectly fixed" beams, as it is seldom possible to "fix" them absolutely for want of the necessary room for the requisite number of rivets, and are, therefore, treated as "supported" beams. Their flanges, consequently, are really stronger than the calculations show, but this is a good provision, because the rolling load, which constitutes nearly the entire load they are called upon to sustain, comes upon them so suddenly.

Load on Longitudinals.

Dead Load.—This consists of—

		lbs.
5·5 feet run of steel rails, at 90 lbs. per yard run, including fastenings, ...	=	165
5·5 feet run of timber 12″ × 6″, at 40 lbs. per cubic foot,	=	110
About 25 feet superficial of ¼″ plate, at 10 lbs. per foot superficial, ...	=	250
An average of 4 inches of ballast, at 93 lbs. per cubic foot, ...	=	775
Total, exclusive of weight of longitudinals themselves,	=	1,300

* Unwin's "Wrought-Iron Bridges and Roofs," 1869, page 64.

Hence, the distributed dead load = 0·58 ton + weight of longitudinals.

Live Load.—This consists of the weight on one driving wheel, say $7\frac{1}{2}$ tons (being equivalent to a distributed load of 15 tons) applied as a concentrated single load, producing the greatest direct stresses when it is at the centre of the longitudinal, and the greatest shearing stress when it is just clear of the cross beam, either coming on or going off the longitudinal.

Working Resistances.—As the load is almost entirely *live* we shall have $s_t = 3·5$ and $s_c = 2·75$ tons (para. 307, Vol. I.), and since from paras. 360 and 361, Vol. I., it would appear that the shearing and bearing strength of wrought-iron is much the same as the tensile strength, except that the former should be somewhat reduced in the case of "group" rivetting, if we take $s_b = 3·5$ and $s_s = 3$ tons we shall be making sufficient allowance for the sudden imposition of the load.

Estimated Weight of Longitudinal.

The weight of the longitudinal is approximately estimated by Professor Unwin's formula,* as given in para. 279, Vol. I., Eq. (15).

Let W = weight of longitudinal in tons, and W_1 = the equivalent distributed load on the girder.

* Professor Unwin's formula is arrived at as follows :—

It is known by experience that from $\frac{2}{3}$ to $\frac{3}{4}$ the weight of a girder is in the booms (Unwin's Wrought-Iron Bridges and Roofs, p. 40) ; also the volume of a well designed boom should be proportional to its length and to the area of its section at the centre.

Let W = weight of main girder ; W_1 = total external distributed load in tons, (*i.e.*, exclusive, of course, of girder's own weight), w = weight of metal per unit volume ; A = mean gross area of both booms together in square inches ; s' = working stress intensity at centre ; r = ratio of clear span to "effective" depth.

Then $\frac{2}{3}\,W = A \times L \times w$, and $A = \frac{mW}{L}$, where m is a constant

Also, Bending Moment at centre $= \frac{(W + W_1)}{8}L$, and moment of resistance $= s'A\,\frac{D'}{2}$.

$\therefore \frac{1}{8}(W + W_1)\,L = \frac{1}{2}\,s'AD'$, and $(W + W_1) = \frac{4\,s'AD'}{L}$.

Substituting for A, as above, we have $(W + W_1) = \frac{4\,s'D'm\,W}{L^2}$, whence $W = \frac{W_1\,L^2}{CD's' - L^2}$, where C is a constant. Dividing numerator and denominator by D′ we have $W = \frac{W_1\,Lr}{Cs' - Lr}$.

In order to deduce the value of C from girders of known weight, we have from above relation

$$C = \frac{(W + W_1)\,L^2}{WD's'} = 4\,\frac{AL}{W},\ \text{since}\ W + W_1 = \frac{4\,s'\,AD'}{L}.$$

Then $W_1 = 0{\cdot}58 + 15 = 15{\cdot}58$ tons (exclusive, of course, of girder's own weight).

L = clear span in feet $= 5{\cdot}5$, and D' = effective depth $= 1$ foot.

Then $r = L \div D' = 5{\cdot}5$.

$C = 1{,}500$, and s' = mean working stress intensity on areas of both flanges in tons per square inch $= \frac{1}{2}(3{\cdot}5 + 2{\cdot}75) = 3$ tons per square inch about.

$$\text{Then } W = \frac{W_1 Lr}{Cs' - Lr} = \frac{15{\cdot}58 \times 5{\cdot}5 \times 5{\cdot}5}{1500 \times 3 - 5{\cdot}5 \times 5{\cdot}5} = 0{\cdot}105 \text{ ton.}$$

Hence total distributed load $= 15{\cdot}58 + 0{\cdot}105 = 15{\cdot}69$ tons about, and the intensity of load per square inch $= 15{\cdot}69 \div 5{\cdot}5 \times 12 = 0{\cdot}24$ ton about.

Flanges.

For the *tension flange*, we have for the bending moment at the centre (para. 182, Vol. I., Ex. 8) $\frac{Wl}{8} = A_t s_t d$, all quantities being expressed in inches and tons.

$$\text{Whence } A_t = \frac{Wl}{8 \times s_t \times d} = \frac{15{\cdot}69 \times (5{\cdot}5 \times 12)}{8 \times 3{\cdot}5 \times 12} = 3{\cdot}08 \text{ square inches.}$$

For the *compression flange*. Since $\frac{s_t}{s_c} = \frac{3{\cdot}5}{2{\cdot}75}$, we have, since by para. 190, Vol. I.,

$$A_c : A_t :: s_t : s_c,$$

$$\therefore A_c = \frac{3{\cdot}5}{2{\cdot}75} \times 3{\cdot}08 = 3{\cdot}9 \text{ square inches.}$$

Forming the flanges out of two angle-irons, each $2\frac{1}{2}'' \times 2\frac{1}{2}'' \times \frac{7}{16}''$, rivetted to the web, and assuming the rivets at $\frac{3}{4}''$ diameter, we have

$$A_t = 2 \times \{2 \times 2\tfrac{1}{2}'' - \tfrac{7}{16}'' - \tfrac{3}{4}''\} \times \tfrac{7}{16}'' = 3{\cdot}3 \text{ square inches.}$$

$$A_c = 2 \times \{2 \times 2\tfrac{1}{2}'' - \tfrac{7}{16}''\} \times \tfrac{7}{16}'' = 4 \text{ square inches.}$$

Hence, the flanges can be made of same section, *vide Fig.* 257, *Plate* XLIII.

Thickness of web and pitch of web rivets.

The greatest shearing stress occurs at the end of the girder and amounts to $\frac{1}{2} \times 15{\cdot}69$ tons $= 7{\cdot}85$ tons, when the driving wheel of the engine is either just coming on or rolling off. The web must be everywhere thick enough to—(1) give a sufficient bearing area on the rivets

connecting it with the flanges; (2) resist buckling without incurring the cost of too many stiffeners.

The shearing stress per foot in height of the web at the piers = reaction at the pier $\div$ height of web in feet = 7·85 tons $\div$ 1 foot (since the effective depth of the web may, in this case, be taken for its full height). The horizontal shearing stress along the joints of the web with the flanges will, therefore, also be 7·85 tons per foot run.

Assuming the rivets to be $\frac{3}{4}''$ diameter, as before, we have, taking the working resistance of the iron in bearing at 3·5 tons per square inch, the pitch at 2″, giving 6 rivets per foot run, and τ as required thickness of plate for web.*

$$6 \times \tau \times \tfrac{3}{4}'' \times 3{\cdot}5 \text{ tons} = 7{\cdot}85 \text{ tons, whence } \tau = \tfrac{1}{2} \text{ inch.}$$

The shearing strength will be more than sufficient.

Web Stiffeners.

The clear distance between the angle-irons is 7 inches. The shear per foot run of web = 7·85 tons, which, since the web is $\frac{1}{2}''$ thick, and there are therefore 6 square inches in its section, is at the rate of 7·85 $\div$ 6 = 1·3 tons per square inch.

Substituting in Eq. (8), para. 237, Vol. I., we have

$$\frac{s_c}{1 + 2_c\left(\frac{d''}{t}\right)^2} = \frac{2{\cdot}75 \text{ tons}}{1 + \frac{2 \times 49 \times 4}{3000}} = 2{\cdot}43 \text{ tons,}$$

which is much greater than 1·3 tons, the load actually to be carried.

Deflection of Longitudinals.

It will be sufficient to determine the maximum permissible deflection by the formula of para. 286, Vol. I., on the supposition that the beam is supported at its ends and loaded at the centre. Taking $f_t \div s$ = 3·5 $\times$ 2,240 lbs. and $f_c \div s$ = 2·75 $\times$ 2,240 lbs., we have

$$\delta = \frac{n''}{\mathrm{E}_t}\left(\frac{f_t + f_c}{s}\right)\frac{c^2}{d} = \tfrac{1}{3} \times \frac{6{\cdot}25 \times 2240}{18000000} \times \frac{33^2}{12} = 0{\cdot}02 \text{ inch.}$$

This would be reduced by one-half if the ends of the girder were "fixed."

Weight of Longitudinals.

The section and elevation of the beam may now be drawn as in *Fig.* 257 and its weight calculated.

* A pitch of 3 inches would be preferable, and the Student should exercise himself with a view to the adoption of this pitch, in the necessary alteration in the design.

With regard to *Fig.* 257 it may be remarked that it would make sounder work if covers were put at A and A instead of cranking the angle-iron, as the weld in the latter method is liable to injure the iron.

The full depth of the longitudinals is $13\frac{1}{2}$ inches, the extra $1\frac{1}{2}$ inches being due to the distance from the centre of figure of the flanges to the outsides of the beam. This extra depth will not necessitate recalculating the web for resistance to buckling, as it was found to be more than the required strength.

The weight is as follows (vide *Molesworth's Pocket-Book*):—

		lbs.
Two lengths of 13 feet each of $2\frac{1}{2}'' \times 2\frac{1}{2}'' \times \frac{7}{16}''$ angle-iron $= 26 \times 6{\cdot}65$ lbs., ...	$=$	173
Plate iron for web $= 5{\cdot}5' \times \frac{13{\cdot}5}{12}$ of $\frac{1}{2}''$ iron $= \frac{5{\cdot}5 \times 13{\cdot}5 \times 20 \text{ lbs.}}{12}$	$=$	124
(6×13) rivets, diameter $\frac{3}{4}''$, length $(\frac{22}{16}'' + 2 \times \frac{5}{4} \times \frac{3}{4}'')$ $= 78 \times \frac{52}{16 \times 12} \times 1{\cdot}476$ lbs., ...	$=$	31
Total weight one longitudinal,	$=$	328

Hence, weight of each longitudinal $= \frac{328}{2240} = 0{\cdot}146$ ton. This is more than the estimated weight ($0{\cdot}105$ ton), but the difference is not sufficiently great to render recalculation necessary.

The Cross Beams.

These beams may be either in the condition of beams "supported" at the ends, "imperfectly fixed," or "fixed," according to the manner in which they are connected with the main girders. If they merely rest on the lower flange, without having their ends rivetted to the web or stiffeners, of the main girder, they are nearly free to bend as a supported beam; but this arrangement is unusual, although it strains the stiffener of the main girder less, and is, therefore, better for the main girder (*Fig.* 263). It is more common to see them rivetted firmly to the web and stiffeners of the main girder (*Fig.* 264, *Plate* XLIV.), and sometimes the connection is made more complete by gusset pieces (*Fig.* 265). The last case (*i.e.*, with gussets) only differs from that of perfect fixation in that the stiffeners are possibly liable to bend laterally, but this cannot be to any great extent. If the top flanges of the cross beams are rivetted to the bottom main flange, or the bottom cross beam flange to the top main

flange (*Figs.* 266 and 267) they become perfectly "fixed" beams. The former arrangement (*Fig.* 266), however, is objectionable, as the sudden stress comes chiefly on the inner rivets at *m*, which would be liable to yield.

It is usual to design cross girders as if "supported only," unless, of course, they are really "fixed," as above explained. In this example calculations will be shown for each case.

The Cross Beams, supposed "supported."

The load to be provided for consists of four equal loads symmetrically placed, two on each side of the centre, each consisting of the weight on one driving wheel of the engine, in addition to the load on one longitudinal and the uniform load due to the weight of the cross beam itself, since each cross beam carries four longitudinals, and each longitudinal carries one line of rail (*see Plate* XLIII.)

Working Resistances.—These will be taken the same as for the longitudinals, for reasons which will become apparent later on.

Estimated Weight of Cross Beam.

In estimating the weight of the cross beams it will be sufficient to reduce the weights above enumerated to an uniformly distributed load, by equating the maximum moment of flexure at the centre of the beam, due to the four isolated loads, with that due to an uniformly distributed load, the beam being, as already explained, supposed to be supported freely at the ends.

It is easy to see that the maximum bending moment produced by the four equal loads W' placed symmetrically at distances of x_1 and x_2 from either end of the cross beam (*Fig.* 271) is equal to $W' (x_1 + x_2)$.

Also that a uniformly distributed load, amounting to W_1 in all, would produce a maximum bending moment $= \frac{1}{8} W_1 l$ (Eq. 44, para. 182, Vol. I.)

Equating these two values for maximum bending moment, we have

$$W_1 = 8 W' (x_1 + x_2) \div l.$$

Now, $W' = 8{\cdot}2$ tons (being made up of $7{\cdot}5$ tons weight on driving wheel $+ 0{\cdot}58$ ton permanent load on longitudinal $+ 0{\cdot}146$ ton, weight of longitudinal itself), $x_1 = 5'$, $x_2 = 10'$, and $l = 26'$.

Whence $$W_1 = \frac{8 \times 8{\cdot}2 \times 15}{26} = 37{\cdot}8 \text{ tons.}$$

Now, the estimated weight of the cross beam is by Unwin's formula (Eq. 15, para. 297, Vol. I.)

$$= W_1 \, Lr \div (Cs' - Lr).$$

It is proposed to make the depth of the girder $= \frac{1}{12}$th span $= 26$ inches (the span being 26 feet), so that $r = L \div D' = 12$. Also $W_1 =$ 37·8 tons, $L = 26$ feet, $C = 1{,}500$, $s' = \frac{1}{2}(3{\cdot}5 + 2{\cdot}75) = 3{\cdot}11$ tons.

$$\therefore \text{ estimated weight} = \frac{37{\cdot}8 \times 26 \times 12}{1500 \times 3{\cdot}11 - 26 \times 12} = 2{\cdot}19 \text{ tons.}$$

Therefore, total live load amounts to $2 \times 7{\cdot}5 = 15$ tons, and total dead load to $\{2 \times (0{\cdot}58 + 0{\cdot}146) + 2{\cdot}19\} = 3{\cdot}65$ tons, so that proportion of live to dead load is about 5 to 1.

Flanges.

It is proposed to build the cross girder of the section shown in *Fig.* 224*a*, *Plate* XL., the web being secured to the flanges by means of angle-irons, and two or more plates being used for each flange.

In cases where great accuracy is required, the diagram of bending moments must be drawn separately for the uniformly distributed load as well as for the imposed loads, but in the present instance, as the weight of the girder bears such a small proportion to that of the imposed loads, it will be sufficiently accurate to combine the two, as shown in *Plate* XL., *Figs.* 217 and 218, which show the diagram of bending moments for the weights combined.

Assuming the angle-irons to be $3\frac{3}{4}'' \times 3\frac{3}{4}'' \times \frac{1}{2}''$ and the rivets, therefore, to be $1\frac{1}{2}'' \times \frac{1}{2}'' = \frac{3}{4}''$ diameter, two rows in each angle-iron and two in each plate, the effective section of the pair of angle-irons required for the tension flange will be $= 2 \times (3\frac{3}{4}'' + 3\frac{3}{4}'' - \frac{1}{2}'' - 2 \times \frac{3}{4}'') \times \frac{1}{2}'' = \frac{11}{2}$ square inches; that of those for the compression flange $= 2 \times (2 \times 3\frac{3}{4}'' - \frac{1}{2}) \times \frac{1}{2} = 7$ square inches.

The moment of resistance of the angle-irons in tension, therefore $=$ $\frac{26}{12}''$ (depth of girder) $\times \frac{11}{2}$ square inches (area of angle-irons) $\times 3{\cdot}5$ tons $= 41{\cdot}7$ foot-tons; that of those in compression $= \frac{26}{12}'' \times 7$ square inches $\times 2{\cdot}75$ tons $= 41{\cdot}7$ foot-tons, the same as for the tension flange.

As the longest ordinate CC′ of *Fig.* 218 measures 6·5 lineal (or 26 load) units and the pole distance 20 load (or 5 lineal) units, the maximum bending moment measures $6{\cdot}5 \times 20$ (or 26×5) $= 130$ foot-tons. Consequently the moment of resistance to be supplied by the plates of each flange $= 130 - 41{\cdot}7 = 88{\cdot}3$ foot-tons.

Taking the width of the girder at $\frac{1}{40}$th the span about, (para. 195, Vol. I.), the flange width will be 8 inches, and the total thickness t of plates required for the respective flanges at the centre of the girder given by the relation—

For the tension flange—$(8'' - 2 \times \frac{3}{4}'') \times t \times 3{\cdot}5$ tons $\times \frac{28}{12}'' = 88{\cdot}3$ foot-tons, whence $t = 1{\cdot}8$ inches.

For the compression flange—$8'' \times t \times 2{\cdot}75$ tons $\times \frac{28}{12}'' = 88{\cdot}3$ foot-tons, whence $t = 1{\cdot}6$ inches.

Each flange, therefore, may be made up of four $\frac{1}{2}''$ plates.

To ascertain the necessary length of the plates, the *Third Method* of para. 193, Vol. I., may be employed. Set off CD, *Fig.* 218, *Plate* XL., to represent 41·7 foot-tons, that is, measure $CD = \frac{41{\cdot}7}{20} = 2$ lineal units, and draw DF parallel to A′C. Then DC′ represents the moment of resistance to be supplied by the plates of each flange at the centre of the girder, and the ordinate between DF and the curve C′F that to be furnished at any given section. Since there are to be four plates, divide DC′ into four equal parts in the points G, K and L, and through them draw straight lines parallel to A′C and meeting the curve in M, N and H. Then it is clear that the uppermost plate may be dropped at H, about 6 feet either side the centre of the girder, and the other plates at N, M and F respectively, although it will be practically advisable to continue the last plate the whole length of the girder instead of dropping it at F.

The Design of the Web.

Figs. 220 and 224, *Plate* XL., show the state of shearing stress, the former when both lines of way are fully loaded, the latter when only the left one is so. The maximum stress evidently occurs at each end of the girder when both lines of way are fully loaded, and measures 17·5 tons, the stresses diminishing in value towards the centre, at intervals of 5 feet; at the centre it never exceeds a value of about 5 tons, *Fig.* 224.

The web must be thick enough (1) to give sufficient bearing surface to the rivets, and (2) to resist any tendency to buckle.

If $\frac{3}{4}''$ rivets be used, as for the flanges, taking 6 per foot run, or a pitch of 2 inches, we have for the stress at the piers an intensity of $17{\cdot}5 \div \frac{28}{12}$ ton per foot of web height, since the height of the web is practically

26 inches, and this amount, therefore, will also measure the intensity of shear per foot run of the *length* as well as the height of the web, (para. 234, Vol. I.); so that, if t be the thickness required for bearing area at the piers, we have for one foot in height or length of the web,

$$6 \times t \times \tfrac{3}{4}'' \times 3\cdot 5 \text{ tons} = 17\cdot 5 \text{ tons} \div \tfrac{26}{12}'', \text{ whence } t = 0\cdot 5 \text{ inch.}$$

Fig. 224 shows the maximum shear at the centre to measure 5 tons. Hence, for the thickness t of the web at the centre we have as just explained

$$6 \times t \times \tfrac{3}{4}'' \times 3\cdot 5 \text{ tons} = 5 \text{ tons} \div \tfrac{26}{12}'', \text{ whence } t = 0\cdot 15 \text{ inch.}$$

As, however, it is undesirable to use plates of a less thickness than $\frac{5}{16}''$ for the webs of open air girders (para. 475), we shall make the web $\frac{1}{2}''$ thick for a distance of 5 feet from each end, and $\frac{5}{16}''$ thick for the remaining length. The web will, therefore, be built up of three plates, two $\frac{1}{2}''$ thick for the end lengths of 5 feet, and one $\frac{5}{16}''$ thick for the intermediate length.*

Web Stiffeners.—Of the 26 inches depth of web available for resisting the shearing stress, a width of $3\frac{3}{4}$ inches both at top and bottom is stiffened by the angle-irons, so that a depth of $18\frac{1}{2}$ inches only remains which is liable to buckle. While, therefore, a sectional area of $26'' \times \frac{1}{2}'' = 13$ square inches is available for resisting the shearing force F, we can only reckon on a depth of $18\frac{1}{2}$ inches for resistance to buckling.

Substituting, therefore, in Equation (8), para. 237, Vol. I., we have on the right side of the equation, for the actual intensity of stress to be provided for at the centre of the girder,

$$\frac{F}{dt} = \frac{17\cdot 5 \text{ tons}}{13 \text{ square inches}} = 1\cdot 34 \text{ tons per square inch,}$$

and, for the maximum permissible stress intensity, putting $s_c = 2\cdot 75$ tons, and $d' = 18\cdot 5$ inches, we have on the left side of the equation—

$$\frac{s_c}{1 + 2c\left(\frac{d'}{t'}\right)^2} = \frac{2\cdot 75}{1 + \frac{2}{3000} \times \left(\frac{18\cdot 5}{0\cdot 5}\right)^2} = 1\cdot 44 \text{ tons per square inch.}$$

Therefore, one $\frac{1}{2}''$ plate is sufficient without any stiffeners for the end portions of the web.

Now, it is evident from *Fig.* 220, that the maximum shearing stress on the central web plate occurs at 3 feet either side the centre when both lines of way are simultaneously loaded, and that it then amounts

* A pitch of 3 inches would be preferable, and the Student should exercise himself in the consequent alterations in the design.

to 8·75 tons. The height of the web being practically 26 inches and the thickness of this portion $\frac{5}{16}''$, its sectional area is 8·13 inches, and the intensity of stress, therefore, about 1 ton per square inch. If $\frac{l}{d} = 26'' \div \frac{5}{16}''$ be substituted for $\frac{d'}{t'} = \frac{18{\cdot}5''}{0{\cdot}5''}$ in the second equation above, it will be seen that the maximum permissible intensity of stress amounts to rather over 1 ton per square inch, so that no stiffeners are required for the central portion of the web, the $\frac{5}{16}''$ plate being sufficient of itself.

Section and Elevation of Cross Girders.

The section and elevation of the cross girders can now be drawn in, *Fig.* 224*a*.

In order to keep the pieces of plate-iron within the limits of length and weight charged for at ordinary market rates, it will be convenient to arrange the lengths as shown in *Fig.* 218, with a grouped joint at the place indicated by a firm line.

Thus, each bottom and top plate will be in one length of 11 feet 6 inches. The remaining plates will each be in three lengths. That next to the bottom or top plate of the pile will have its middle portion 7 feet 4 inches long, and each outer portion 4 feet 4 inches long. The third plate from top or bottom will have its middle portion 8 feet, and each end portion 5 feet 6 inches long. The fourth or inner plate of the upper flange will consist of three portions each 8 feet 8 inches long. The fourth or inner plate of the lower flange will be 8 feet 8 inches between the joints, and have outer portions each 10 feet 11 inches long, which latter will be turned up at their outer ends to meet the ends of the corresponding plate of the upper flange. *See* also *Fig.* 224*a*.

Joints in the Flanges.

There will be two joints, one on each side the centre of the girder, as shown in *Fig.* 218.

The rivet diameter being $\frac{3}{4}''$, the pitch will $= (2\frac{1}{2}$ to $3) \times \frac{3}{4}'' = 2''$ say.

The thickness of covers might be $1\frac{1}{2}'' \times \frac{1}{2}'' = \frac{3}{4}''$, but will be made $\frac{1}{2}''$ like the plates.

The strength of each plate in the tension flange $= (8'' - 2 \times \frac{3}{4}'')$

$\times \frac{1}{2}'' \times 3{\cdot}5$ tons $= 11{\cdot}38$ tons; that of each compression plate $= 8'' \times \frac{1}{2}'' \times 2{\cdot}75$ tons $= 11$ tons.

Now, only three of the four plates are jointed, the remaining one acting merely as a distance piece. Hence, taking the strength of each at 11 tons, we have, since $m = 2$,

For *Condition* 3° (para. 477), 11 tons $\times 3 = N'' \times 0{\cdot}78 \times (\frac{3}{4}'')^2 \times 3$ tons, whence $N = 25$, or say 26.

Condition 4°, 11 tons $\times 3 = 2\,(8'' - 2 \times \frac{3}{4}'') \times \frac{1}{2}'' \times 3{\cdot}5$ tons $+ n'' \times 2 \times \frac{3}{4}'' \times \frac{1}{2}'' \times 3{\cdot}5$ tons, whence $n'' = 3{\cdot}9$, or say 4, and since $N = 2\,(n' + n'') = 26$, we have $n' = 9$, or say 10.

Hence, the length of cover plates $= 2'4''$, and the joint may be drawn in as indicated.

Angle Covers.

No angle covers will be required, as it is easy to get angle-irons 26 feet long, and there will be no joint except over the supports where a cover is not necessary.

Rivets to connect Angle-irons to Flange Plates.

The greatest intensity of horizontal shear occurs over the supports where it amounts, as already explained, to $17{\cdot}5 \div \frac{26}{12} = 8$ tons about per foot run, so that for single shear of the rivets, if N'' be the number required, we have

$$N'' \times \frac{\pi}{4} \times (\tfrac{3}{4}'')^2 \times 3 \text{ tons} = 8 \text{ tons, whence } N'' = 6 \text{ rivets,}$$

and for the number N' required for bearing, we have

$$N' \times \tfrac{3}{4}'' \times \tfrac{1}{2}'' \times 3{\cdot}5 \text{ tons} = 8 \text{ tons, whence } N' = 6 \text{ rivets,}$$

and as there are two rows, three per foot run, or a pitch of 4 inches, will suffice.

Covers to Web Plates.

As the web consists of three plates, one at each end $\frac{1}{2}''$ thick, and one central plate $\frac{5}{16}''$ thick, the simplest arrangement will be to make the plates flush on one face and use a single $\frac{5}{16}''$ cover to each of the two joints, thus avoiding any packing pieces except on one side between the angle-irons and central web plate. Were the web liable to much exposure to the weather, double covers would be desirable so as to exclude wet from the joints.

The number of rivets required for shearing and bearing at the ends of the web has been determined above, the rivets being the same throughout, (*viz.*, $\frac{3}{4}''$), and the central portion of the web being only $\frac{5}{16}''$ thick with a maximum intensity of stress of $8{\cdot}75 \div \frac{26}{12} = 4$ tons per foot run, it will be found that 3 rivets per foot run are sufficient in shearing and 5 in bearing. A pitch of 2 inches, therefore, will be adopted, giving 6 rivets per foot run on either side of the joint.

The joints in the web plates will, therefore, have single covers of $\frac{5}{16}''$ metal, the rivets having a pitch of 2 inches, which allows of 9 on each side the joint, as the distance between the angle-irons is $18\frac{1}{2}$ inches.

The length of covers from top to bottom of web will be $18\frac{1}{2}$ inches, and their width, to allow of no rivet being nearer the edge of either the cover or the plates than the distance laid down in para. 484 (8), amounting to $4 \times (\frac{1}{2}'' + \text{thickness of plate} + \frac{1}{2}d) = 4 \times (\frac{1}{2} + \frac{5}{16}'' + \frac{1}{2} \times \frac{3}{4}'') = 4\frac{3}{4}''$, say 5, inches, to allow a margin, for the direction of the fibres will be across the cover plate, and on one side each rivet hole there will be the edge of the cover plate and on the other that of one of the plates to be joined.

Deflection of Cross Beams.

The deflection of a beam very similar to the one under consideration is examined in the Addendum to Chapter VIII. At the time the Addendum was printed, the section of the cross girder was designed as in *Fig.* 35*a*, but owing to subsequent changes made in the values of the working resistances, the design has been correspondingly altered. There will, in consequence, be a slight difference in the deflection due to the new value of I, the Moment of Inertia of the cross section about an axis passing through its centre of figure. This in the former case was taken = 6,250 inch-units; the new value of I will be found = 9,000 units about (para. 489), and as in the Addendum referred to it is shown that the deflection of beams, varying only in the Moments of Inertia of their cross sections, varies inversely as the values of those Moments of Inertia, we have for the deflection δ' of the beam with revised cross section, seeing that of the former was = 0·34 inch,

$$\delta' = 0{\cdot}34 \times \frac{6250}{9000} = 0{\cdot}24 \text{ inch},$$

so that the ratio of $\frac{\delta'}{l} = 0{\cdot}24 \div 26' \times 12 = \frac{1}{1300}$, which is well within the limit laid down by Professor Rankine (*see* para. 282, Vol. I.)

Weight of Cross Beams.

The cross girder can now be drawn and its weight calculated as follows :—

I.—$\frac{1}{2}''$ *plate, at 20 lbs. per foot superficial for flanges and web.*

$$\left.\begin{array}{l} \left.\begin{array}{l} \text{Outer plates} = 2 \times 11\tfrac{1}{2}' \times \tfrac{8}{12}{}' \\ \text{Second ,,} = 2 \times 16' \times \tfrac{8}{12}{}' \\ \text{Third ,,} = 2 \times 19' \times \tfrac{8}{12}{}' \\ \text{Inner ,,} = (26' + 30\tfrac{1}{2}') \times \tfrac{8}{12}{}' \end{array}\right\} = 149\tfrac{1}{2}' \times \tfrac{8}{12}{}' \\ \text{Covers, } 2 \times 2' \times 17{\cdot}5 \div 12 \\ \angle \text{ irons, } 2 \times 56{\cdot}5' \times (2 \times 3{\cdot}5 - 0{\cdot}5) \div 12 \\ \text{Outer web plates, } 2 \times 5 \times 26 \div 12 \end{array}\right\} = 188{\cdot}3 \text{ feet superficial, at 20 lbs.} = 3{,}766 \text{ lbs.}$$

II.—$\frac{5}{16}''$ *plate, at 12·5 lbs. per foot superficial for web.*

$$\left.\begin{array}{l} \text{Central web plate} = 1 \times 8\tfrac{2}{3} \times (26 \div 12) \\ \text{Web covers, } 2 \times (18{\cdot}5 \div 12) \times (5 \div 12) \end{array}\right\} = 20{\cdot}1 \text{ feet superficial, at 12·5 lbs.} = 251$$

III.—*Packing.* $\frac{2}{16}''$ *plate, at 5 lbs. per foot superficial.*

$2 \times 10 \times (3{\cdot}75 \div 12) = 6{\cdot}25$ foot superficial, $=$ 31

IV.—*Rivets.* $\frac{3}{4}''$ *round iron, at 1·5 lbs. per foot run.*

$$\left.\begin{array}{l} \text{Web row in } \angle \text{ iron, at 2'' pitch} = (2 \times 57 \times 6) \times (1\tfrac{1}{2} + 1\tfrac{1}{4} \times \tfrac{3}{4}) \times \tfrac{1}{12}, \\ \text{Flange row, at 2'' pitch} = (2 \times 57 \times 6) \times (2\tfrac{1}{2} + 1\tfrac{1}{4} \times \tfrac{3}{4}) \times \tfrac{1}{12}, \\ \text{Web covers, at 2'' pitch} = (1\tfrac{1}{2} \times 6) \times (2\tfrac{5}{16} + 1\tfrac{1}{4} \times \tfrac{3}{4}) \times \tfrac{1}{12}, \end{array}\right\} = 364 \text{ feet, at 1·5 lb.} = 546$$

Total, ... 4,594

Thus each Cross Beam weighs $\frac{4594}{2240} = 2{\cdot}05$ tons, or almost exactly the estimated weight.

Connecting Longitudinals with Cross Beams.

The number of rivets required to connect the longitudinals with the cross beams may be determined as follows :—

They have to resist a total shear of $\frac{17{\cdot}5}{2} = 8{\cdot}75$ tons, so that, using $\frac{3}{4}''$ rivets, if N″ be the total number required to resist single shear, we have—

$N'' \times \left\{ \frac{\pi}{4} \times (\frac{3}{4}'')^2 \times 3 \text{ tons} \right\} = 8{\cdot}75$ tons, whence $N'' = 6{\cdot}6$ or 7 rivets, say.

For bearing $N' \times \frac{3}{4}'' \times \frac{1}{2}'' \times 3{\cdot}5$ tons $= 8\ 75$ tons, whence $N' = 6{\cdot}6$ or 7 rivets, say.

Hence, 4 rivets in each angle-iron, or 8 in all, will be amply sufficient.

The Cross Beam, considered as "Fixed."

This is dealt with in Example II., Chapter XXVI, wherein the bending moment and deflection polygons are determined. *See Plate* XL.

Flanges —It will be seen from *Fig* 217 that the bending moment over the supports amounts to 90 foot-tons; that at the centre of the girder to 40 foot-tons.

If the same angle-irons are employed as for the beam, supposed "supported," *viz.*, $3\frac{3}{4}'' \times 3\frac{3}{4}'' \times \frac{1}{2}''$, the moment of resistance of which has been shown to amount to 41·7 foot-tons, whether in compression or tension, it is evident that no flange plates at all are required except for a short distance of about 3 feet at either end of the beam, the thickness of which for the tension flange over the supports is given by the relation—

$$(8'' - 2 \times \tfrac{3}{4}'') \times t \times 3{\cdot}5 \text{ tons} \times \frac{26''}{12} = (90 - 41{\cdot}7) = 48{\cdot}3 \text{ foot-tons},$$

whence $t = 0\ 93$ inch; and for the compression flange over supports by $8'' \times t \times 2{\cdot}75$ tons $\times \frac{26''}{12} = 48{\cdot}3$, whence $t = 1$ inch.

Hence, it will be sufficient if a half-inch plate extends to a length of 3 feet from each end of the girder, both on tension and compression flanges, and an additional half-inch plate be secured to it for a distance of 2 feet from either end *See Fig.* 224*b*, *Plate* XL.

Web.—The *design of the web* will be the same as for the "supported" beam (para. 463).

Deflection —The *deflection* is given by the formula arrived at in para. 119, *viz.*, $y = y' \times \frac{pl^3}{6EI}$.

From para. 489 we have

$$I = \tfrac{1}{12}\{8'' \times (26'')^3 - 2\,(3{\cdot}25'' \times 25''^3 + 0{\cdot}5'' \times 18{\cdot}5^3)\}$$
$$= 2{,}726 \text{ inch-units of inertia.}$$

$l = 26 \times 12$, $p = 20$ tons $= 20 \times 2{,}240$ lbs., $E = 18{,}000{,}000$, and y' measures 9 inches about, being the deflection shown on *Fig.* 217.

$$\text{Hence } y = \frac{9 \times (20 \times 2240) \times (26 \times 12)^3}{6 \times 18000000 \times 2726} = 0{\cdot}14 \text{ inch about.}$$

The Main Girders.

The Main Girders, 70 feet long, carry the platform, the rolling load, the longitudinals and cross beams, and their own weight. In the case of the cross girders, the approximate weight was determined by means of Professor Unwin's formula, as given in Eq. 15, para. 279, Vol. I. It being unnecessary to again illustrate the use of this formula, we shall at once turn to Sir B. Baker's Tables of gross weights of Bridges, given in Appendix E. Therein the weight of the two main girders of a double line Plate Iron Bridge, of 70 feet span, the cross girders of which are secured to the lower flanges of the main girders, is fixed at 8 8 cwt. per foot run.

The load, therefore, to be borne by the main girders, is as follows:—

Load to be carried by Main Girders.

Permanent load to be carried by two main girders—		tons.
Weight of 44 longitudinals $= 44 \times 0{\cdot}14$ ton	=	6·16
" platform on longitudinals $= 44 \times 0{\cdot}58$	=	25·52
" 11 cross girders $= 11 \times 1{\cdot}98$ ...	=	21·78
" side platforms $= 2{\cdot}5'$ (width) $\times 2 \times 66'$ (length) $\times$ 41 lbs. ...	=	6·04
" two main girders $= \dfrac{8{\cdot}8 \text{ cwt.} \times 70 \text{ feet}}{20}$	=	30·80
Total permanent load on two main girders ...	=	90·30
(1). Dead load, therefore, to be carried by one main girder	=	45·15
(2). Live load on one girder = rolling load on one set of rails $= 66 \times 1\frac{1}{2}$ tons	=	99 00
∴ Total weight to be carried on one main girder	=	144·15

The proportion of live to dead load is, therefore, about 2 to 1, and the Working Resistances will, by para. 397, Vol. I., be

Working Resistances — $s_t = 4{\cdot}2$ tons, $s_c = 3{\cdot}3$ tons, $s_b = 4{\cdot}2$ tons, $s_s = 3$ tons.

The Design of the Main Girders.

Of the above loads, the weight of the girder itself is evidently the only distributed load, the remaining loads being applied at the points in

which the 11 cross girders are secured to the main girders and at the supports. As the former load constitutes such a small proportion of the whole, being less than one-ninth of it, it will be sufficient to combine all the loads together, and suppose the whole load applied at the points of junction referred to.

The total load, therefore, of 144 tons will be supposed apportioned at the 11 points in which the cross girders meet the main girders, together with half a portion applied at each support. There will be 12 tons at each point of juncture and 6 tons at each support.

Fig. 261 shows the stress and shearing diagrams under this hypothesis, and *Fig.* 259 the bending moment diagram. The deflection polygon is shown in *Fig.* 260.

As a pole distance of 76 load (or 19 lineal) units has been employed, and the maximum ordinate of the bending moment polygon measures 15 lineal (or 60 load) units, the magnitude of the maximum bending moment evidently $= 76 \times 15$ (or 19×60) $= 1{,}140$ foot-tons. The maximum shearing force at the supports measures 66 tons.

We shall assume the depth of the girder to be $\frac{1}{12}$ span $= \frac{66'}{12} = 5{\cdot}5$ feet, and the angle irons to be $4\frac{1}{2}'' \times 4\frac{1}{2}'' \times \frac{5}{8}''$, so that the diameter of the rivets will be $= 1\frac{1}{2} \times \frac{5}{8}'' = \frac{15}{16}''$, or say 1 inch. The width of the flanges will be taken at $\frac{1}{33}$ the span $= 2$ feet, (para. 195, Vol. I.), and we shall suppose there are four rows of rivets running the length of each flange.

The effective section of the angle-irons in tension will be $= 2 \times (2 \times 4\frac{1}{2}'' - \frac{5}{8}'' - 2 \times 1'') \times \frac{5}{8}'' = \frac{255}{32}$ square inches, and their moment of resistance $= \frac{255}{32}$ square inches $\times$ 4·2 tons $\times$ 5·5 feet (depth of girder) $= 184$ foot-tons.

The effective section in compression will be $2 \times (2 \times 4\frac{1}{2}'' - \frac{5}{8}'') \times \frac{5}{8}'' = \frac{335}{32}$ square inches, and the moment of resistance $= \frac{335}{32}$ square inches $\times$ 3·3 tons $\times$ 5·5 feet $= 190$ foot-tons.

If t be the total thickness of plates required in either case, we shall have—

For the tension flange. $(24'' - 4 \times 1'') \times t \times 4{\cdot}2$ tons $\times$ 5·5 feet $= 1140 - 184 = 956$ foot-tons, whence $t = 2{\cdot}1$ inches.

For the compression flange. $24'' \times t \times 3{\cdot}3$ tons $\times$ 5·5 feet $= 1140 - 190 = 950$ foot-tons, whence $t = 2{\cdot}2$ inches.

Hence, each flange may be built up of four $\frac{5}{8}''$ plates, 24 inches wide.

To ascertain the points at which each plate may be dropped, we set off vertically below A′C, *Fig.* 259, after Method 3°, as explained in para. 193, Vol. I., a distance representing 184 foot-tons (the tension flange angle-irons being the weaker), that is, since the pole distance measures 76 load units, we set down $CD = \frac{184}{76} = 2\frac{4}{10}$ lineal units to represent the resistance of the angle-irons, and draw DE parallel to CA′; then will the ordinates between DE and the curve of bending moments measure, at any point, the moment of resistance that must be supplied by the plates. As the number of plates required for each flange is four, by dividing the distance DC′ into four equal parts and drawing straight lines through them, the points at which the several plates may be dropped are at once determined. These are the points H, K and L.

The Design of the Web.

From *Fig.* 261 it is seen that the total shearing stress at the piers = 66 tons, so that the intensity of stress per foot in height of web = 66 ÷ 5·5 feet = 12 tons per foot run = intensity of stress also per foot run horizontally of web, (para. 234, Vol. I.)

Taking a trial thickness of $\frac{1}{16}''$ for the web at the piers and rivets of 1″ as before, we have for the total number N′ of rivets required per foot run for bearing, the expression $N' \times \frac{1}{16}'' \times 1'' \times 4\ 2$ tons = 12 tons, whence N′ = 45·7 rivets. Hence, a thickness of $\frac{12}{16} = \frac{3}{4}''$ for the web plate at the piers will necessitate $\frac{46}{12}$, or say 4 rivets per foot run, that is, a pitch of 3 inches.

This number will be found more than sufficient for requirements of shearing.

Towards the centre of the girder the shearing force diminishes, and the web can consequently be correspondingly reduced in thickness, provided it be everywhere maintained thick enough to afford sufficient bearing area and resist buckling.

By drawing the scale *ff*, shown to the left of *Fig.* 261 and dividing it into 12 equal parts, each part may be taken to represent a web thickness of $\frac{1}{16}''$ and the points where the several plates may be dropped might thereby be at once determined, always remembering that a less thickness than $\frac{5}{16}''$ is undesirable, but as the cross girders are rivetted

to the main girders at distances of 5 feet 6 inches it will be convenient to reduce the web thickness at these points.

Thus, the thickness *might*, it would seem, be reduced at intervals of 11 feet as follows, provided the thicknesses proposed are sufficient to prevent buckling, which it will be seen however they are not, *viz.*:—

For a distance of 11 feet from each support web thickness might be $\frac{3}{4}''$.

From 11 feet to 22 feet from each support web thickness might be $\frac{1}{2}''$.

From 22 feet from either support to girder's centre web thickness might be $\frac{5}{16}''$.

Web stiffeners.—In order to determine the strength of the web to resist buckling, it will be necessary, while bearing in mind the limits laid down in para. 399, Vol. I., as to length, weight, breadth, &c., of plate-iron, to apply formula (8*a*), para 238 of that Volume, and it will then be found advisable to have stiffeners at less intervals than even 5 feet 6 inches, if the thickness of the web is not to be unnecessarily increased. Moreover, the limit of breadth mentioned in para. 399 referred to is 4 feet. Let, then, the thickness of the web be reduced at intervals of 5 feet 6 inches, and additional stiffeners introduced midway, that is, at intervals of 2 feet 9 inches.

If a $\frac{5}{16}''$ plate be the thinnest used, it will require rivets of $(\frac{3}{2} \times \frac{5}{16}'' =)\ \frac{15}{32}$, or say $\frac{1}{2}''$ diameter, necessitating cover plates having a width of at least $4 \times (2 \times \frac{1}{2}'') + 2 \times \frac{1}{2}'' = 5$ inches, or say 6 inches, at the junction, so as to allow a distance of two diameters between the circumference of each rivet hole and the edge of the plate or cover plate as laid down in para. 484 (8). This 6 inches width of cover plate will stiffen the web and thereby reduce the "effective length" of the hypothetical pillar, referred to in para. 399, Vol. I., to $\sqrt{2}\,(2'\ 9'' - 6'') = \sqrt{2} \times 27''$.

Taking, then, in Equation (8*a*), para. 238, Vol. I., $s_c = 3{\cdot}3$ tons, $d' = 66''$, $d'' = 27''$ and $c = 3{,}000$, the expression reduces itself to the convenient form

$$20{\cdot}58\, t^2\, (217{\cdot}8\, t - F) = F,$$

by substituting in which, approximate values of t (varying by $\frac{1}{16}''$ in value) can be determined.

In this way, the following thicknesses at intervals of 5 feet 6 inches

will be found sufficient (bearing in mind, that is, that one intermediate stiffener will be provided in each interval):—

From each extremity of the girder to a distance of
5′ 6″ from end cross girder a thickness of $\frac{12}{16} = \frac{3}{4}''$.
At from 5′ 6″ to 11′ from each support a thickness of $\frac{10}{16} = \frac{5}{8}''$.
" " 11′ to 16′ 6″ " " " $\frac{9}{16}''$.
" " 16′ 6″ to 22′ " " " $\frac{8}{16} = \frac{1}{2}''$.
" " 22′ to 27′ 6″ " " " $\frac{6}{16} = \frac{3}{8}''$.
" " 27′ 6″ to girder centre " " $\frac{5}{16}''$.

This is shown in *Fig.* 255, *Plate* XLIII.

The main stiffeners, *viz.*, those at 5 feet 6 inches intervals, assist in securing the cross beams which rest on the bottom flange, and in transmitting their load to the web.

It is usual to make stiffeners either of angle or tee-irons, in order to stiffen the whole girder, and as they should be 6 inches wide at least, as already explained, the main stiffeners might be made of tee-iron, 6″ × 3″ × $\frac{3}{8}''$, which is a market section, and the intermediate ones of plate-iron, each of the thickness of the web at the joint they cover.

The stiffener over each support must be treated as a pillar, which has to resist a compression equal to the shearing force at that point. If, therefore, the least dimension be made rather over $\frac{1}{10}$ the height $= \frac{66}{10} = 6{\cdot}6$ inches, say $6\frac{3}{4}$ inches to 7, the pillar may be treated as a "short column," and in that case will require a sectional area $= \frac{\text{shearing force}}{\text{safe stress intensity}} = \frac{66 \text{ tons}}{3{\cdot}3 \text{ tons}} = 20$ square inches, which may be obtained by using two tee-irons 6″ × 3″ × $\frac{3}{4}''$, one rivetted each side the web, as shown in *Fig.* 268, *Plate* XLIV. It would, however, be better still to introduce a gusset piece, as shown in *Fig.* 269.

As the girder will be built with a camber, and as, when it deflects under the load, the position of the centre of the bearings alters, it will be well to introduce an additional stiffener at the inner end of the bearing, as shown in *Fig.* 255, *Plate* XLIII.

Joints in Flanges.

The limiting weight of pieces of plate-iron, priced at ordinary market rates, is fixed at 4 cwt. (*vide* para. 399, Vol. I.) As a length of 8

feet 11 inches of plate-iron 2 feet wide and $\frac{5}{8}''$ thick weighs about 4 cwt. this length should not be exceeded, if prices are to be kept within ordinary rates. It will be found that this condition is observed in the grouping of joints shown in firm lines in *Fig.* 259, necessitating four joints X, Y, Z and V on each side the girder's centre. That at V might, of course, be dispensed with, and would be so in practice, but it is retained in order to show the method of calculation.

The cross section of each plate being $24'' \times \frac{5}{8}''$ with four longitudinal rows of $1''$ rivets, the maximum permissible stress in a plate of the tension flange will be $(24'' - 4 \times 1'') \times \frac{5}{8}'' \times 4{\cdot}2$ tons $= 52{\cdot}5$ tons, and in the compression flange $= 24'' \times \frac{5}{8}'' \times 3{\cdot}3$ tons $= 49{\cdot}5$ tons. Therefore it will be sufficient to deal generally with the tension flange. We shall have, then, (referring to para. 477) $\sigma = 52{\cdot}5$ tons, and $T' = 52{\cdot}5 \times 4$ tons.

Thickness of cover plates. Least thickness $= \left(\frac{\nu}{\nu + 1}\right) \tau = \frac{4}{5} \times \frac{5}{8}'' = \frac{1}{2}''$, but a thickness of $\frac{5}{8}''$ (same as plates) will be adopted, being more convenient.

JOINTS AT X AND Y. Here $\nu = 4$. *Condition* 1° is already satisfied.

Condition 2° (bearing). We have $N' = \dfrac{2\,\sigma}{d \tau s_b} = \dfrac{2 \times 52{\cdot}5}{1 \times \frac{5}{8} \times 4{\cdot}2} = 40$ rivets.

This value of N', therefore, will be constant throughout.

Condition 3° (shearing). $N'' = \dfrac{T'}{0{\cdot}78 \times d^2 \times s_s} = \dfrac{52{\cdot}5 \times 4}{0{\cdot}78 \times 1' \times 3} = 89{\cdot}7$, or say 92 rivets.

Condition 4°. $n''(\nu - 1) \times 0{\cdot}78 \times d^2 \times s_s = T' - 2\,(b - md)\,\tau' s_t$

$$\therefore n'' = \frac{105}{3 \times 0{\cdot}78 \times 3} = 14{\cdot}96, \text{ or say 16 rivets.}$$

Hence, since $N = 2n' + 3n''$ we have

$$n' = \tfrac{1}{2}\,(92 - 48) = 22, \text{ or say 24 rivets.}$$

Condition 5°. Substituting these values in the expression

$$\left\{2n'\,\tau' + \frac{\nu}{2}\,(\nu - 1)\,n''\,\tau\right\} d s_b$$

we obtain the value $(2 \times 24 + 2 \times 3 \times 16) \times \frac{5}{8} \times 1 \times 4{\cdot}2 = 378$ tons, which is $1{\cdot}8$ times T' ($= 210$ tons).

Hence corrected value of $N = 2n' + 3n'' = 2 \times 24 + 3 \times 16 = 96$ rivets.

The flanges may, therefore, be designed as indicated.

With a pitch of 3 inches the length of cover plate will be $(96 \div 4) \times (3 \div 12) = 6$ feet.

JOINT AT Z. Here $n = 3$. Hence new value of $N = \frac{3}{4} \times$ old value $= \frac{3}{4} \times 96 = 72$ rivets.

Substituting in equation (12) of para 479 we have for new value n''

$$n'' = 16 \times \tfrac{3}{2} - \tfrac{1}{2} \times \frac{52 \cdot 5}{4 \cdot 2} \times \frac{8}{5} = 14 \text{ rivets.}$$

Therefore $n' = \frac{1}{2}(N - 2n'') = \frac{1}{2}(72 - 28) = 22$ rivets.

And with a 3″ pitch the length of each cover plate $= (72 \div 4) \times (3 \div 12) = 4\frac{1}{2}$ feet.

JOINT V. In this case $\nu = 1$, and in dealing with this joint it will be well to bear in mind the following, *viz* :—(1), that in comparing the value of N obtained for this joint under para. 477 with that of n which would be obtained under para 367, Vol. I., the former represents the *total* number of rivets required for the joint, whereas n only represents the number required on one side of the joint, so that $N = 2n$; and (2), that the values of N obtained for the joints X, Y and Z have been fixed by the requirements for shearing strength only.

Comparing, therefore, the new value of N (for shearing) when $\nu = 1$, with its value ($= 96$) when $\nu = 4$, we have the relation $N_1 = \frac{1}{4} \times 96 = 24$, whereas the value of N' for *bearing* has already been shown to be constant throughout the flange and $= 40$.

Joint V, therefore, will require 40 rivets, 20 being placed each side the joint, so that with a pitch of 3 inches the length of cover plate $= (40 \div 4) \times (3 \div 12) = 2\frac{1}{2}$ feet.

Pitch of Rivets. The pitch $= (2\frac{1}{2}$ to $3)$, $d = 3$ inches say.

The several joints may now be drawn in.

Angle Covers and Angle-iron Joints.

The effective section of one angle-iron $4\frac{1}{2}'' \times 4\frac{1}{2}'' \times \frac{5}{8}''$ has been already shown to be $= \frac{1}{2} \times 7 \cdot 97 = 3 \cdot 99$ square inches, and its available strength (in the tension flange) $= 3 \cdot 99$ square inches $\times\ 4 \cdot 2$ tons $=$ 16 76 tons.

Whence, the number of rivets N'' required for shearing is given by the relation, $N'' \times 0 \cdot 78 \times 1'' \times 3$ tons $= 16 \cdot 76$ tons, whence $N'' = 7 \cdot 1$, say 8, and for bearing by the relation $N' \times 1'' \times \frac{5}{8}'' \times 4 \cdot 2$ tons $= 16 \cdot 76$ tons, whence $N' = 6$ 4. Hence 8 rivets are required, 4 in each arm of the angle-iron.

The cross section of the cover must be equal to that of the angle-iron, and the cover may be of $3\frac{7}{8}'' \times 3\frac{7}{8}'' \times \frac{3}{4}''$ angle-iron.

The position of the joints in the angle-iron is unimportant.

They should not, however, occur where the angle cover will interfere with the putting together of the rest of the work, and they may break joint with one another, though some Engineers prefer to bring the joints together for convenience of surveillance.

Camber and Deflection.

Fig. 260 shows the Deflection Curve of this Main Girder, the maximum ordinate of which measures 9 lineal units $= 9 \times 12$ inches. Substituting in the equation, para. 119, we have

$$y = y' \times \frac{pl^2}{6 \times E \times I}$$

in which $y' = 9 \times 12$ inches, $p = 76 \times 2{,}210$ lbs., $l = 66 \times 12$ inches, and $E = 18{,}000{,}000$ lbs., we obtain the following value for the maximum deflection, *viz.* :—

$$y = 0{\cdot}697 \text{ inch.}$$

Allowing one-third of this for permanent set, we have the true deflection $= 0{\cdot}46$ inch, which, being divided in direct proportion to the dead and live load, gives $0{\cdot}15$ inch for the former and $0{\cdot}3$ inch for the latter, or $\frac{1}{2640}$ of the span for the live load only.

Weight of the Main Girder.

The weight of the girder may now be taken out under the following heads. It will be observed that the method adopted is slightly different to that employed in the case of the cross girders.

$$\begin{array}{lll}
\textit{Flanges.} & \text{Outer} = 2 \times 30{\cdot}5' & = 61 \\
 & \text{Second} = 2 \times 43' & = 86 \\
 & \text{Third} = 2 \times 54' & = 108 \\
 & \text{Fourth} = 2 \times (70' + 6') & = 152 \\
\textit{Covers.} & \text{Joints X and Y} = 2 \times 4 \times 6 & = 48 \\
 & \text{Joint Z} = 2 \times 2 \times 4{\cdot}5 & = 18 \\
 & \text{Joint N} = 2 \times 2 \times 2{\cdot}5 & = 10
\end{array} \left. \vphantom{\begin{array}{l}1\\1\\1\\1\\1\\1\\1\end{array}} \right\} = 483' \times 2' \times \frac{5'}{8 \times 12} \quad \overset{\text{c.ft.}}{= 50{\cdot}31}$$

Angle-iron. $4 \times 76' \times \frac{1}{12} (2 \times 4\frac{1}{2} - \frac{5}{8}) \times \frac{5}{8 \times 12}$ $= 11{\cdot}06$

Carried forward, ... $61{\cdot}37$

		c.ft.
	Brought forward, ...	61·37
Angle Covers.	$2 \times 6 \times \frac{30}{12} \times \frac{1}{12}(2 \times 3\frac{7}{8} - \frac{3}{4}) \times \frac{3}{4 \times 12}$	$=$ 1·09
Web.	$2 \times 5{\cdot}5' \times 5{\cdot}5' \times \frac{1}{12}(\frac{5}{16} + \frac{3}{8} + \frac{1}{2} + \frac{9}{16} + \frac{5}{8})$ $+ 2 \times 7' \times 5{\cdot}5' \times \frac{3'}{4 \times 12}$	$=$ 16·78
Main Stiffeners $=$ (T-iron 6″ × 3″ × ⅜″)	$2 \times 11 \times 7' \times \frac{1}{12}(6 + 3 - \frac{3}{8})$ $\times \frac{3}{8 \times 12}$	$=$ 3·46
Intermediate.	$2 \times 12 \times 5{\cdot}5' \times \frac{6}{12} \times \frac{17}{32 \times 12}$ (mean thickness)	$=$ 2·92
End Stiffeners (T-iron 6″ × 3″ × ¾″)	$2 \times 2 \times 7' \times \frac{1}{12}(6 + 3 - \frac{3}{4})$ $\times \frac{3}{4 \times 12}$	$=$ 1·20

Rivets.

$$\left.\begin{array}{ll}
\textit{Flanges}\text{—}16 \times (2 \times 75{\cdot}5) \times \frac{1}{12}(\frac{10}{8} + \frac{10}{8})\text{ (mean)} & = 500 \\
\left.\begin{array}{l}
\textit{Angle-irons (web)}\text{—}(2 \times 75{\cdot}5 \times 4) \times \frac{1}{12}(\frac{3}{8} + \frac{10}{8}) \\
\textit{Web Stiffeners and Covers}\text{—}(27 \times 7') \times (4 \times 2) \times \frac{1}{12}(\frac{3}{8} + \frac{10}{8})
\end{array}\right\} & = 191
\end{array}\right\} 691 \times \frac{\pi}{4} \times \left(\frac{1}{12}\right)^2 = 3{\cdot}78$$

Total, $=$ 90·60

Weight $= 90{\cdot}6$ cubic feet $\times \frac{484 \text{ lbs.}}{112} = 391{\cdot}5$ cwt.

Hence the weight per foot run $= \frac{391{\cdot}5}{70} = 5{\cdot}6$ cwt.

The estimated weight, as given by Sir B. Baker's Tables, was 4·4 cwt.* about, so that the actual weight exceeds the estimated weight of the whole girder by about 4 tons.

Total weight of girder $= 19\frac{1}{4}$ tons about (instead of about $15\frac{1}{2}$ tons).

If accuracy in the result is desired, the girder should be re-calculated.

The ends of the girder may be rounded off as shown in *Fig.* 255.

Expansion arrangements. These have already been dealt with.

* These Tables give fairly correct results for the usual values of working resistances. When special values are employed, as in this case, the amounts given in the Tables should be correspondingly altered.

EXAMPLE III.*

THE DESIGN OF A BOX GIRDER.

It is required to design a box girder of the form shown in *Plate* XLV. The girder is to be of uniform section, to replace a partition wall under the gutter of the M roof shown in *Fig.* 277.

Its clear span is 30 feet and width 14 inches. Its effective depth is to be as small as possible, consistently with the girder fulfilling the two following conditions :—

1st, that the working resistance shall not exceed—

In tension	5 tons	per square inch.
,, compression	4 ,,	
,, shearing	5 ,,	
On bearing surfaces	7 ,,	

2nd, that the deflection shall not exceed $\frac{3}{4}''$ under a uniformly distributed vertical load (including the weight of the girder) of 14 tons.

NOTE.—The shearing and bearing resistances are taken higher than usual, the load being dead, and the deflection not being influenced materially by them. The resistances to tension and compression might be higher were it not that a very stiff beam is required.

The girders will rest at each end on a brick wall 14 inches thick, a stone template being placed under each end to distribute the weight over a sufficient area.

The total length of the girder will be $30' + 2 \times 14'' = 32$ feet 4 inches, and the span, from centre to centre of bearings, may be taken at 31 feet.

It is not proposed to "fix" the ends of the girders.

Calculations.

Depth necessary to fulfil the two given conditions.

This being a beam of uniform section, uniformly loaded, with a cross section of equal strength, the maximum permissible deflection is given by the formula (Eq. 8*e*, para. 286, Vol. I.) :—

$$\delta = \frac{n''}{E_t}\left(\frac{f_t + f_c}{s}\right) \cdot \frac{c^2}{d}, \text{ whence } d \text{ may be found.}$$

* This Example is taken, with slight alterations, from Wray and Seddons' "Instruction in Construction," p. 211, 3rd Edition.

In this case $\delta = \frac{3}{4}''$ or about $\frac{1}{40}$ inch per foot run.

$$n'' = \frac{5}{12} \text{ (para. 286, Vol. I.)}$$

$$\frac{f_t}{s} = 5 \times 2{,}240 \text{ lbs. and } \frac{f_c}{s} = 4 \times 2{,}240 \text{ lbs.}$$

$$c = \text{half span} = \tfrac{1}{2} \times 31' \times 12 = 186 \text{ inches.}$$

$E_t = 18{,}500{,}000$ lbs. = the Modulus of Elasticity of plate beams is taken high, as the number of joints will not be large (Rankine's "Civil Engineering," page 531).

d = required effective depth in inches.

$$\text{Then } \tfrac{3}{4} = \frac{5}{12}\left(\frac{9 \times 2240}{18{,}500{,}000}\right)\frac{186 \times 186}{d}.$$

$$\therefore d = 20{\cdot}9 = 21 \text{ inches.}$$

Section of Flanges.

The section of greatest stress being at the centre, and the beam being of uniform section, it is sufficient to determine the dimensions of the flanges at the centre.

On the assumption that the whole of the direct stresses are taken by the flanges, and the whole shearing stress by the web, and the load uniformly distributed; at the centre—(paras. 182 and 189, Vol I.)

$$M_m = \frac{W\,l}{8} = s_t\, A_t\, d, \text{ or } s_c\, A_c\, d, \text{ Moment of Resistance.}$$

In this case W = 14 tons distributed.

" " w per inch run $= \dfrac{14}{12 \times 31} = \dfrac{7}{186}$.

" " $d = 21''$.

" " the values of l, s_t, s_c are as given above.

Whence A_t = *effective* section of tension flange = 6·2 square inches.

A_c = *gross* section of compression flange = $\frac{5}{4}$ A_t = 7·75 square inches.

Each flange may be taken to consist of one plate and two angle-irons (*Fig.* 274), the plate being weakened by three rivets in its cross section (the centre rivet occurring at the cover-plates), and the angle-irons by two rivets each, one in each arm (*Figs.* 274 and 276).

It is apparent, from the small sectional area required, that one plate will be enough, and it is as well (not necessary) to have the angle-irons and plate the same thickness.

Assume the angle-iron to be $3'' \times 3''$, which will be sufficient to allow of the angle covers being rivetted on, and call the thickness of plate and angle-irons, τ. Then it is clear that τ will be $< \frac{1}{2}''$, since the flange plates are to be 14 inches wide, and the maximum area of one plate and two angle-irons $= 7{\cdot}75$ inches only; hence, the diameter of the rivets should be 2τ.

For the *tension flange*—

The effective section $A_t = (14'' - 3 \times 2\tau)\,\tau$ (for plate) $+ 2\,(6'' - \tau - 2 \times 2\tau)\,\tau$ (for angle-irons) $= 6{\cdot}2$ square inches.*

$$\text{Whence, the thickness of the plate and angles} = \tau = \tfrac{5}{16}'',$$
$$\text{and the diameter of the rivet holes} = 2\tau = \tfrac{5}{8}''.$$

The loss of strength in the plate, due to the rivet holes, is about 13 per cent.; the loss of strength on the whole flange is about 17 per cent.

For the *compression flange*—

The effective, which is the gross, section $A_c = 11'' \times \tau$ (for plate) $+ 2\,(6'' - \tau)\,\tau$ (for angle-irons) $= 7{\cdot}75$ square inches.

Whence $\tau = {\cdot}3'' = \frac{5}{16}''$ say.

Cover plates to Flanges.

In order to keep the plates within the length (15 feet), for which no extra change is made, let there be three lengths of plate in each flange, the joints in each plate being thus about 5 feet on each side of centre of girder.

The stress on one side of the joint is first taken by the rivets, by them transferred to the cover plate, which in turn transfers it to the rivets on the other side of the joint, and so to the plate on that side.

The load being distributed at any section of the girder—

the flange stress $\times$ depth of girder $= \dfrac{wx'\,(l - x')}{2}$ (equation 43, para. 182, Vol. I.)

Therefore at 5 feet from the centre of the girder—

$$\text{the flange stress} = \frac{wx'\,(l - x')}{2d} = \frac{\frac{7}{186} \times 126\,(372 - 126)}{2 \times 21} = 27{\cdot}8 \text{ tons.}$$

For the *tension flange*—

effective section of plate $= (14'' - 3 \times \frac{5}{8}'') \times \frac{5}{16}'' = \frac{485}{128}$ square inches,

* If the rivetting be zigzag, and the rivets widely spaced, only two would be deducted from the plate, and one from each angle-iron, to obtain the effective section; but such an arrangement need not be considered in this case, as it would not do to use thinner plates and angles than $\frac{5}{16}''$, though the angles might be reduced to $2\frac{1}{2}'' \times 2\frac{1}{2}''$.

effective section of angle-irons $= 2\left((6'' - \frac{5}{16}'') - 2 \times \frac{5}{8}''\right)\frac{5}{16}'' = \frac{355}{128}''$,
and the effective section of both plate and angles $= \frac{840}{128}$ square inches.

Hence, of the total flange stress of 27·8 tons, the plate, and therefore the cover has to bear $\frac{485}{840}$ of 27·8 tons = 16 tons, nearly.

For the number, n'', of rivets in the plate on each side of the joint, in order to be secure against shearing,

$$n'' \times \frac{\pi}{4} \times (\tfrac{5}{8})^2 \times 5 \text{ tons} = 16 \text{ tons},$$

whence $$n'' = 10{\cdot}4.$$

For bearing surface—

$$n' \times \tfrac{5}{16} \times \tfrac{5}{8} \times 7 \text{ tons} = 16 \text{ tons},$$

whence $$n' = 11{\cdot}7,$$

so that 12 rivets must be adopted.*

Let the pitch of the rivets in these plates be taken at 8 inches. Then, as there will be four rows on each side of the joint, the cover plates will be 24 inches long, and it will be seen afterwards that these rivets range with the other rivets in the work. As there is only one cover, it had better be $\frac{6}{16}''$ thick. It would be far better construction to have two covers, but as they could not be less than $\frac{1}{4}''$ thick, it would take a little more metal.

For the *compression flange*—

$$\left.\begin{array}{l}\text{the gross section of the plate} = 14'' \times \frac{5}{16}'' = \frac{70}{16} = \frac{560}{128}'' \\ \text{and the gross section of the angle-irons} = 2 \times \frac{455}{256} = \frac{455}{128}\end{array}\right\} = \frac{1015}{128}''$$

The plate has on it $\frac{560}{1015}$ of 27·8 tons = 15·3 tons.

It was found above that $\frac{5}{8}''$ rivets in $\frac{5}{16}''$ plates are weaker in bearing than shearing, hence—

For bearing surface, $n' \times \frac{5}{16} \times \frac{5}{8} \times 7$ tons = 15·3 tons,
whence $n' = 11{\cdot}1$ nearly; or say 12 rivets, as for tension flange.

The cover plates in each flange may therefore be as in *Fig.* 276, subject to alteration, if it be found necessary, in order to make their rivets range with those in the rest of the work.

Angle covers might be got rid of altogether, angle-iron being easily obtainable of the length of this girder (32 feet 4 inches); but, for the sake of showing the calculations, it will be assumed that the angle-irons are jointed at the centre of the girder.

* From Table, para. 480, it is seen at once that with $\frac{5}{8}''$ rivets in $\frac{5}{16}''$ plates, more rivets would be required for bearing than for shearing.

In the *tension flange*, the joint being at the centre of the girder, the relative effective section of the plate and angle-irons being as given above, and the stress at the centre being 6·2 square inches $\times$ 5 tons $=$ 31 tons, the angle-irons have on them $\frac{355}{840}$ of 31 $=$ 13·1 tons.

For shearing, $n'' \times \frac{\pi}{4} \times (\frac{5}{8})^2 \times 5$ tons $= \frac{13 \cdot 1}{2}$ tons, there being two angle-irons;

whence, n'', the number of rivets in each angle-iron $= 4 \cdot 3$.

For bearing surface, $n' \times \frac{5}{8} \times \frac{5}{16} \times 7$ tons $= \frac{13 \cdot 1}{2}$ tons,

whence, $n' = 4 \cdot 8$, or 5 rivets.

Three rivets can be placed on each side of the joint, in one arm of the angle-iron, and two in the other; or three in each may be used, if more convenient.

In the *compression flange*, the angle-irons have on them $\frac{455}{1015} \times 31$ tons $=$ 14 tons nearly.

The rivets and angle covers being the same as in the tension flange, more rivets will be required for bearing than for shearing, hence:—

For bearing, $n' \times \frac{5}{8} \times \frac{5}{16} \times 7$ tons $=$ 14 tons,

whence $n' = 5 \cdot 1$,

so that the angle covers may be the same for each flange.

An angle cover 19 inches long, and equal in section to the angle-iron may be used. It may be $2\frac{3}{4}'' \times 2\frac{3}{4}'' \times \frac{3}{8}''$.

Thickness of Web and Pitch of Rivets connecting Web to Flanges.

The shearing stress under the uniform load being greatest at the points of support, and diminishing to *nil* at the centre (Example 8, para. 182, Vol. I.,) the web may possibly admit of being thinned towards the centre, but it must everywhere be thick enough—1*st*, to give a sufficient bearing area on the rivets connecting it to the flanges; 2*nd*, to resist buckling without incurring the cost of too many stiffeners.

NOTE A.—The capacity of the web to resist *shearing* does not require calculating, since even if the rivet holes are deducted (which is not necessary, as the metal could not shear through the holes without shearing the rivets in the holes) the shearing resistance of the metal left between the rivet holes will be greater than the bearing strength of the web on the rivets, since the bearing resistance is never taken as more than 7 tons to the inch, or 1·75 times the shearing resistance, whilst the clear space between any two rivet holes is always more than 1·75 times the diameter of the holes.

NOTE B.—The simplest method of proportioning the number of rivets, required for bearing, to the thickness of the web plates, is to take the initial thickness of plate web at $\frac{1}{16}''$ (as wrought-iron plates vary in thickness by $\frac{1}{16}''$), and to ascertain the number of rivets required per foot run to connect the web to the flanges at the piers, where the shearing stress is greatest ; then to divide the number of rivets, and multiply the initial thickness of web, by whatever number will reduce the rivets sufficiently to give convenient pitch, and a thickness of web between the ordinary limits of $\frac{1}{4}''$ and $\frac{3}{4}''$ as in previous Example.

The same pitch is given to the rivets all along the girder as at the ends for the sake of uniformity; hence there will, if anything, be an excess of bearing strength towards the centre, as any reduction in the thickness of the web would only be in proportion to the reduced shearing stress.

It is usual to find the shearing stress per foot run of the web, assuming it to be uniformly distributed throughout each vertical section, and to calculate the number of rivets on this assumption.

At the piers the shearing stress is a maximum = half the total load on girder = 7 tons, or $3\frac{1}{2}$ tons on each of the webs, which are 21 inches deep, so that in each web the shearing stress per foot in height $= 3\frac{1}{2} \div \frac{21}{12} = 2$ tons.

The horizontal shearing stress at the piers, along the joints of the web with the flanges, will also be 2 tons per foot run, (para. 234, Vol. I.)

Assume $\frac{5}{8}''$ rivets, as before ; and $\frac{1}{16}''$ as the initial thickness of the web ; then equating the bearing strength of n' rivets with the shearing stress per foot run—

$$n' \times \frac{5}{8}'' \times \frac{1}{16}'' \times 7 \text{ tons} = 2 \text{ tons},$$

whence $n' = 7{\cdot}5$, or say 8 rivets per foot run.

Dividing the 8 rivets and multiplying the $\frac{1}{16}''$ thickness of web by 4, gives 2 rivets per foot run, or a pitch of 6 inches, and a $\frac{1}{4}''$ thickness of web at the piers, and therefore throughout the web, as nothing under $\frac{1}{4}''$ is admissible.

The rivets in the web being in single shear, it remains to see if 2 per foot run will give a resistance to shearing = 2 tons.

The shearing resistance of two $\frac{5}{8}''$ rivets $= 2 \times (\frac{5}{8})^2 \times \frac{\pi}{4} \times 5$ tons $= 3$ tons.

Hence, 2 rivets per foot run are more than enough to resist shearing.

Keeping within the limits of ordinary priced plates, the web may be in three lengths, with a vertical joint at 5 feet on each side of the centre.

Covers to Webs.

If a single cover is used at each joint, of the same thickness as the web, *i.e.*, $\frac{1}{4}''$, it will be more than strong enough, as the shearing stress at either joint is only about $\frac{1}{3}$rd its value at the piers.

In order to make the covers lie close to the web, three rivets are necessary on each side of the vertical joint, as shown in *Fig.* 275, and giving them a $6''$ pitch, as the other rivets, they will, for the reason just given, be more than sufficient for both bearing and shearing.

The width of each cover will be such as to allow of the rivet holes being punched both in the plates and cover without being nearer their edges than $1\frac{1}{2}$ times the diameter of the holes; hence the minimum width will be $8 \times \frac{5}{8}'' = 5''$.

Stiffeners to Web.

The shearing stress being greatest at the piers, and the web uniform in thickness, if stiffeners are necessary at all, they will be required at the piers.

The shearing stress on the webs, at the piers, was shown to be 2 tons per foot in height; therefore the stress per inch in height of each web $= \frac{1}{6}$ ton, and as the web is $\frac{1}{4}''$ thick, the stress per square inch $= \frac{4}{6}$ or $\frac{2}{3}$ ton.

It has been already shown that the vertical and horizontal shearing stresses, having equal intensities, may be resolved into two stresses of the same intensity, one a tension, and the other a thrust, acting at angles of 45° with the horizontal, and at right angles to one another, and that the web tends to buckle under the thrust, (para. 234, Vol. I.).

The intensity of the thrust $\frac{P}{A}$ (para. 70, Vol. I.), in tons per square inch, must not exceed what the material can safely bear, *i.e.*, $s_c \div \left(1 + c\frac{l^2}{d^2}\right)$.

In this case $\frac{P}{A} = \frac{2}{3}$ ton; $s_c = 4$ tons; $c = \frac{1}{3000}$; $d = \frac{1}{4}''$; whence $l = 30{\cdot}6''$, measured between the angle-irons, at an angle of 45°.

The vertical distance between the angle-irons being $15''$ (*Fig.* 275), the distance measured at an angle of 45° is $21''$, or less than $30{\cdot}6''$, so that no stiffeners are necessary, (*Fig.* 275).

Order of Rivetting up.

The girder may now be designed, as shown in *Plate* XLV., the parts being so arranged that each can be rivetted up in the following order:—

The lower flange, with angle irons and cover plates, is first rivetted up; then the webs are put together and connected with the lower flange; then the angle-irons for the top flange are rivetted to the webs; and finally, the plates of the top flange are put together and connected with the webs, the last row of rivets being those connecting the top plates with the angle-irons.

Plate XLV. shows the rivets at 3″ pitch in the covers, and 6″ pitch clear of the covers. A 6″ pitch would be too great for work exposed to the weather; nor would it be advisable to use plates under $\frac{5}{16}$″ thick in any but a dry situation. If the pitch were made 4″ throughout, the work would look better, but it would increase the number of rivets, and thereby the amount of work, as well as lengthen the covers.

The centre row of rivets in the flange plates exists only at the covers, and in calculating the centre section, the centre rivet hole might be left out of consideration, there being no cover there, but as the girder is to be a beam of *uniform section*, it is more correct to suppose it to be present.

Fig. 246

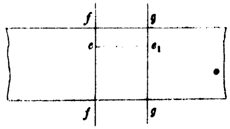


Fig. 247.

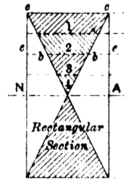


Fig. 248.

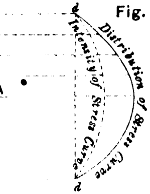


Fig. 249.

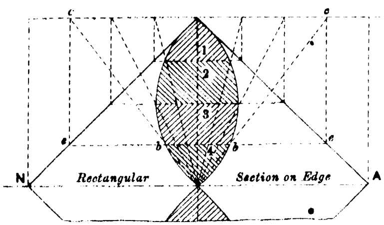


Fig. 250.

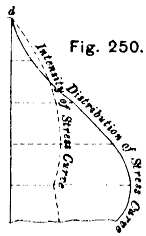


Fig. 251.

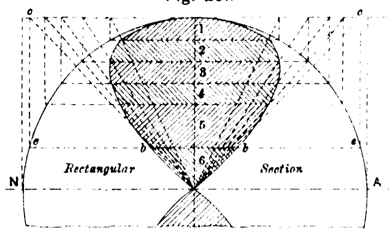


Fig. 252

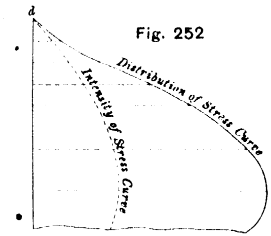


Fig. 253.

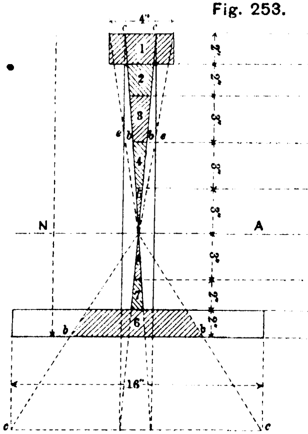
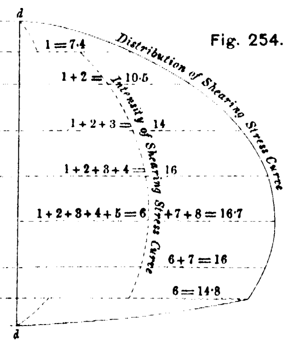
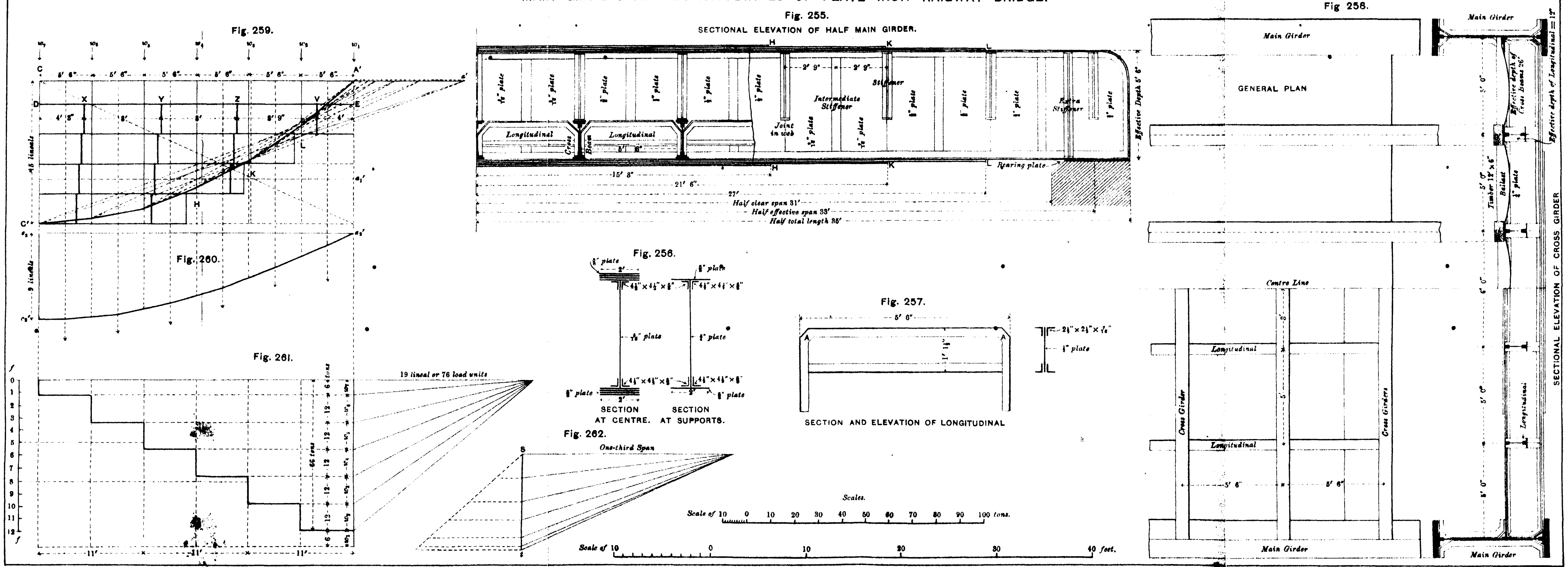


Fig. 254.



Litho. T. C. Press, Roorkee.

MAIN GIRDERS AND LONGITUDINALS OF PLATE-IRON RAILWAY BRIDGE.



Litho., T. C. Press, Roorkee.

Fig. 263.

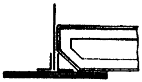


Fig. 264.

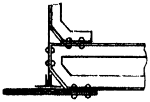


Fig. 265.

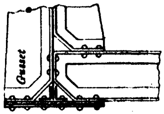


Fig. 266.

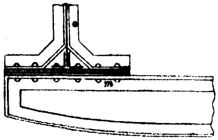


Fig. 267.

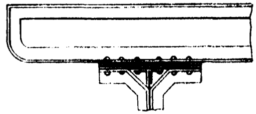
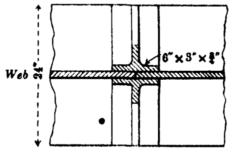
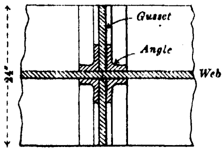


Fig. 268.



SECTIONAL PLAN

Fig. 269.



SECTIONAL PLAN

Fig. 270.

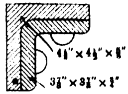


Fig. 271.

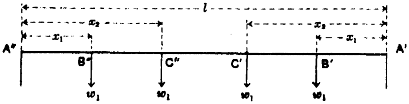
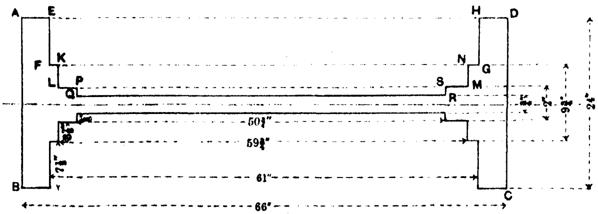


Fig. 272

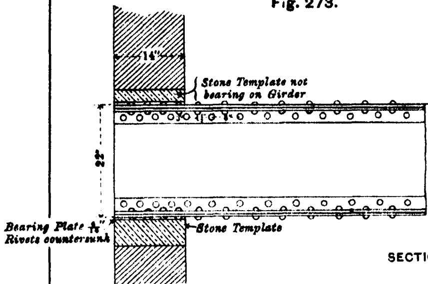


Litho., T. C. Press, Roorkee.

BOX GIRDER.

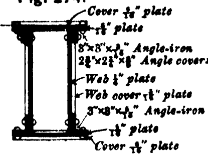
Scale—1/12

Fig. 273.



SIDE ELEVATION.

Fig. 274.



SECTIONAL ELEVATION ON A B C D

Fig. 275.

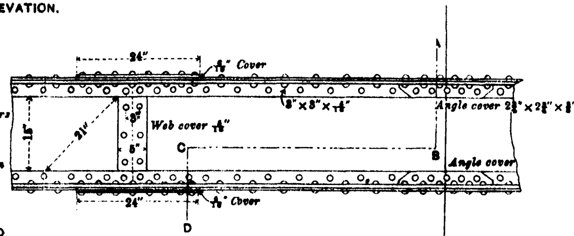


Fig. 276.

PLAN

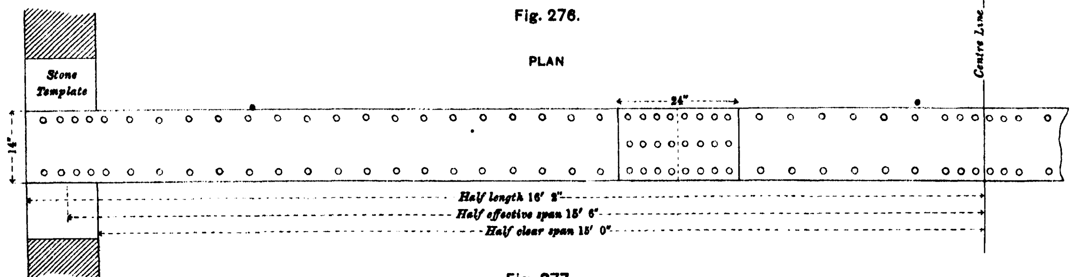
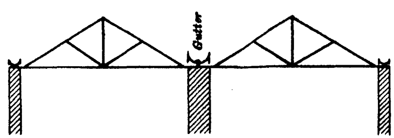


Fig. 277.



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APPENDIX A.

TIED ARCHED ROOFS.

Memo. No. C. $\frac{3032}{878}$ W. A. *of N.-W. P. and Oudh, dated Naini Tal, the* 1*st August,* 1892.

To ensure uniformity in the preparation of drawings and estimates for buildings with tied arch roofs, the following instructions are issued for guidance:—

(1). For spans up to 10 feet, tie-rods to be placed at not more than 5 feet centre to centre.

Cast- or wrought-iron washers of diamond shape $9'' \times 6'' \times \frac{1}{2}''$ can be used for thrust-plates.

Thickness of brick ring to be $4\frac{1}{2}$ inches. Rise to be taken at $\frac{1}{6}$th of span.

(2). When the span exceeds 10 feet and is under 20 feet—

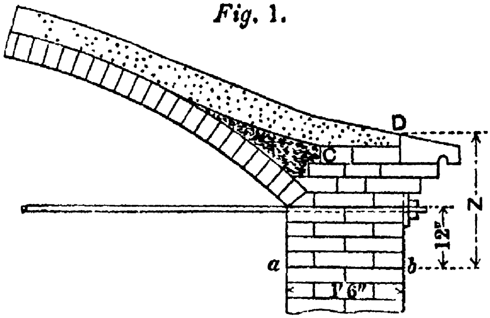
Tie-rods to be not more than 4 feet centre to centre.

Thrust-plates to be of angle-iron (as per accompanying table). Thickness of arch ring to be 9 inches. Rise of arch $\frac{1}{6}$th to $\frac{1}{4}$th of span.

(3). The concrete of the roof should be finished off as shown on the annexed sketch, and should not be brought over to the outer edge of the wall.

Parapets should not be used with tied arch roofs, as they interfere with the free flow of water off the roofs, and the drainage outlets left in them invariably get stopped up.

Fig. 1.



The filling in of the extra thickness at the haunch, shown in a darker shade, should be carried out immediately the centering is removed, and the uniform thickness (of 3 or 4½ inches) of fine terracing should be put on some 20 or 25 days afterwards, so that any slight separation that may occur at C owing to the settlement of the arch would be filled up and covered over by the upper coat of uniform thickness. The weathering course of the cornice should be brick-on-edge, cut to the slope of the roof, set in extra good lime mortar, and plastered with a thin coat of plaster, not more than ¼ inch thick, at the time the upper surface of the terrace roof is being finished off. The backing to this course should be an ordinary course 2¾ inches thick, to allow of a sufficient depth (1¾ inches) at D for the end of the terracing to abut against.

Whatever may be the kind of work in the wall below *ab*, the portion above it marked Z should invariably be set in good lime mortar.

(4). In no case should the arched roof of a room intended for occupation spring from a less height than 12 feet above the floor of the room.

Table showing the dimensions of tie-rods required for several spans of tied arch roofs, also the cost of constructing them per running foot.

Span in feet.	Rise of arch.		Thickness of arch ring.	Diameter of tie-rods when rise $= \frac{S}{6}$.		Diameter of tie-rods when rise $= \frac{S}{4}$.		Dimensions of washers and backing plates.	Cost per running foot when rise $= \frac{S}{6}$.	Cost per running foot when rise $= \frac{S}{4}$.	Remarks
	$\frac{S}{6}$	$\frac{S}{4}$		Calculated.	Practical.	Calculated.	Practical.		Rs. A. P.	Rs. A. P.	
4′	0′—8″	..	$4\frac{1}{2}$″	·42″	$\frac{1}{2}$″	..	..	Washers $9'' \times 4\frac{1}{2}'' \times \frac{3}{4}''$ ..	2 10 9	..	These calculations are based on the following data.—
6′	1′—0″	..	”	·50″	$\frac{5}{8}$″	..	..		3 1 0	..	3″ of concrete to be laid over arch ring.
8′	1′—4″	..	”	·58″	$\frac{3}{4}$″	..	..		3 9 3	..	
10′	1′—8″	..	9″	·78″	$\frac{7}{8}$″	..	..	Continuous thrust plates of angle-iron $3'' \times 3'' \times \frac{3}{8}''$.	7 11 0	..	Tie-rods placed 4′ apart.
12′	2′—0″	3′—0″	”	·85″	$\frac{7}{8}$″	·74″	$\frac{3}{4}$″		8 7 3	8 13 9	Rate of arch masonry at Rs. 30 per cent.
14′	2′—4″	3′—6″	”	·91″	1″	·79″	$\frac{7}{8}$″		9 12 0	10 1 0	Rate of plain masonry at Rs. 22 per cent.
16′	2′—8″	4′—0″	”	·97″	1″	·85″	1″	Continuous thrust plates of angle-iron $4'' \times 4'' \times \frac{3}{8}$.	11 1 9	11 10 9	Rate of concrete at Rs. 15 per cent.
18′	3′—0″	4′—6″	”	1″	$1\frac{1}{8}$″	·89″	1″		12 8 6	13 1 0	Rate of ironwork at Rs. 12 per maund.
20′	3′—4″	5′—0″	”	1″	$1\frac{1}{8}$″	·94″	$1\frac{1}{8}$″		13 7 0	14 0 0	

N.B.—The cost includes one foot of brickwork below tie-rods.

APPENDIX B.

MASONRY DAMS.

Extract from a Report on the Design and Construction of Masonry Dams. By W. J. MACQUORN RANKINE, C.E., LL.D., F.R.S.
[*Published in the "Engineer" of January 5th,* 1872].

Material.—As regards the material best suited for a reservoir wall or embankment, I consider that it must be determined by the nature of the foundation. That foundation should be sound rock, if practicable; and should a rock foundation be unattainable, firm impervious earth. It may be doubted whether any earthen foundation is thoroughly to be relied on where the depth of water exceeds 100 feet or 120 feet. It is not advisable to build a high masonry dam on an earthen foundation; for the base of the dam must be spread to a width sufficient to distribute the pressure, so that it shall not be more intense than the earthen foundation can bear; and this involves the use of a quantity of material which would lead to immoderate expense, if the material used were masonry.

Mode of Building.—In the case of a rock foundation the proper material is unquestionably rubble masonry laid in hydraulic mortar; and the opinion of M. Graeff, that continuous courses in building that masonry are to be avoided, is fully corroborated by experience; for the bed joints of such courses tend to become channels for the leakage of the water.

Precautions.—The very fact, however, of the irregular structure of that masonry, renders necessary unusual care and vigilance in superintending its erection, in order to ensure that every stone shall be thoroughly and firmly bedded, and that there shall be no empty hollows in the interior of the wall, nor spaces filled with mortar alone where stones ought to be placed. The practice of "grouting," or filling hollows by pouring in liquid mortar, should be strictly prohibited. Should it be

resolved to insert in the face of the wall, headers, or long bond-stones, with or without projecting ends to form corbels, as in the dam of the river Furens, those stones ought to be laid with their lengths not horizontal, but normal to the face of the wall.

Principles determining Profile.—With respect to the profile of the wall, its figure is in the main to be determined by principles nearly the same with those laid down by the French Engineers, and put in practice in the dams of the rivers Furens and Ban, that is to say, the intensity of the vertical pressure at the inner face of the wall should at no point exceed a certain limit when the reservoir is empty, and the intensity of the vertical pressure at the outer face of the wall should at no point exceed a certain limit when the reservoir is full.

Limits of Vertical Pressure.—In the theoretical investigation of M. Delocre, and the practical examples given by M. Graeff, the same limit is assigned to the intensity of the vertical pressure at both faces of the wall. But it appears to me that there are the following reasons for adopting a lower limit at the outer than at the inner face. The direction in which the pressure is exerted amongst the particles close to either face of the masonry is necessarily that of a tangent to that face; and, unless the face is vertical, the vertical pressure found by means of the ordinary formula is not the whole pressure, but only its vertical component; and the whole pressure exceeds the vertical pressure in a ratio which becomes the greater, the greater the "batter," or deviation of the face from the vertical. The outer face of the wall has a much greater batter than the inner face; therefore, in order that the masonry of the outer face may not be more severely strained when the reservoir is full than that of the inner face when the reservoir is empty, a lower limit must be taken for the intensity of the vertical pressure at the outer face than at the inner face.

Weight of wall to be thrown inwards.—The proposal of the Executive Engineer to throw the weight of the wall further inwards than in the French designs tends to realise the principles just stated, and so far I fully approve of it and have carried it out in the profile which accompanies this report.*

Wall not to overhang inwards.—I do not, however, concur with the Executive Engineer in the proposal to throw the weight of the wall so

* This profile has not been reproduced.

far inwards as to make it overhang, for the following reason:—The additional stability against the horizontal thrust of the water gained by giving the wall an overhanging batter inwards, is not that due to the whole weight of the overhanging masonry, but only to the excess of that weight above the weight of water which it displaces; in other words, about half the effect of the weight of the overhanging mass of masonry in giving stability is lost through its buoyancy, and hence the additional stability gained by making the wall overhang inwards is not proportionate to the additional load thrown upon the lower parts of the inner face; and more stability would be gained by placing a given mass of masonry, so as to form an uniform addition to the thickness of the wall, than by making it overhang inwards.

Limits of Vertical Pressure, how fixed.—In choosing limits for the intensity of the vertical pressure at the inner and outer faces of the wall represented by the accompanying profile,* I have not attempted to deduce the ratio which those quantities ought to bear to each other from the theory of the distribution of stress in a solid body; for the data on which any such theoretical determination would have to be based are too uncertain. The limits which I have chosen are as follows and they are given in the first place in feet of a vertical column of masonry whose weight would be equivalent to the pressure, and are then reduced to various other measures:—

LIMITS OF VERTICAL PRESSURE AT—

	Inner face.	Outer face.
Feet of masonry, ...	160	125
Feet of water, ...	320	250
Pounds on the square foot (nearly), ..	20,000	15,625
Metres of masonry (nearly), ...	49	38
Metres of water (nearly), ...	98	76
Kilog. on the square centimetre (nearly),	9·8	7·6

In choosing these two limits I have been guided by the consideration of the following facts. As regards the inner face, where the deviation of the direction of the stress from the vertical is unimportant, it is certain, from practical experience, that rubble masonry laid in strong hydraulic mortar, and on good rock foundations, will safely bear a vertical pressure equivalent to the weight of a column of masonry 160 feet high, if not higher. As regards the outer face, the practical data given by M. Graeff show that masonry of the same quality in the sloping outer

* This profile is not reproduced here.

face of a dam will safely bear a pressure whose vertical component, as found by the ordinary rules, is equivalent to the weight of a column 125 feet high.

Diminution of Vertical Pressure towards foot of slope.—The same reasons which show that the intensity of the vertical component of the pressure ought to be less for a battering than for a vertical face show also that this intensity ought gradually to diminish at the lower part of the outer face, where the batter gradually increases. In the present state of our knowledge we should not be warranted in forming any definite theory as to the law which this diminution ought to follow, and therefore, in preparing the accompanying design, I have thought it best to be guided in this, as in the previous case, by practical examples, and to consider it sufficient to make the law of diminution such, that at the depth of 150 feet below the surface the intensity of the vertical component of the pressure at the outer face becomes nearly equal to what it is at the same depth in the outer face of the dam across the Furens, *viz.*, 107 feet of masonry, or about 6½ kilogrammes on the square centimetre.

Tension to be avoided.—I have kept in view another principle, not referred to by the French authors, *viz.*, that there ought to be no practically appreciable tension at any point of the masonry, whether at the outer face when the reservoir is empty, or at the inner face when the reservoir is full. Experience has shown that in structures of brickwork and masonry that are exposed to the overturning action of forces which fluctuate in amount and direction (as when a factory chimney is exposed to the pressure of the wind) the tendency to give way first shows itself at that point at which the tension is greatest. In order that this principle may be fulfilled, the line of resistance should not deviate from the middle of the thickness of the wall to an extent materially exceeding one-sixth of the thickness. In other words, the lines of resistance when the reservoir is empty and full respectively should both lie within, or but a small distance beyond, the middle third of the thickness of the wall.

Horizontal curvature of Wall.—As regards the effect of giving the wall a curvature in plan, convex towards the reservoir, I look upon this as a desirable, and in many cases an essential, precaution, in order to prevent the wall from being bent by the pressure of the water into a curved shape concave towards the water, and thus having its outer face

brought into a state of tension horizontally, which would probably cause the formation of vertical fissures, and perhaps lead to the destruction of the dam. I consider, however, that calculations of stability which treat the dam as a horizontal arch are so uncertain as to be of very doubtful utility; and I would not rely upon them in designing the profile. In fixing the radius of horizontal curvature, I consider that the engineer should be guided by the form of the gorge in which the dam is to be built, making that radius as short as may be consistent with convenience in execution, and with making the ends of the dam abut normally against the sound rock at the sides of the gorge.

Summary of conditions to be fulfilled by Profile.—The conditions which have been observed in designing the accompanying profile may be summed up as follows :—A. The vertical pressure at the inner face not to exceed 160 feet of masonry. B. The vertical pressure at the outer face not to exceed 125 feet of masonry at the point where it is most intense, and to diminish in going down from that point. C. The lines of resistance when the reservoir is full and empty respectively, to lie within or near to the middle third of the thickness of the wall. These are limiting conditions, and do not prescribe exactly any definite form. Any form chosen should fulfil them without any practically important excess in the expenditure of material beyond what is necessary.

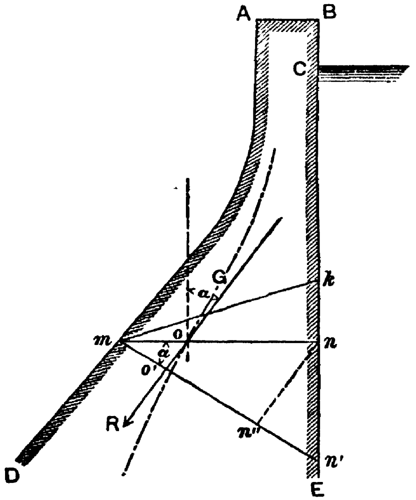
APPENDIX C.

NOTE TO CHAPTER XV.

Note on M. Bouvier's *method of measuring the maximum intensity of stress at a point of the wall's face.*

It is necessary here to refer to a method of calculating the maximum intensity of stress at a point of the wall's face, due to M. Bouvier, and adopted by many Engineers, because the proof offered would seem to rest on unsound premisses. In order to measure the maximum stress acting over the horizontal joint *mn*, *Fig.* 2, the total resultant pressure R on which is represented in direction by *Goo'*, M. Bouvier draws the hypothetical joint *mn'* through *m*, perpendicular to the direction of R, making it intersect the straight line *nn''* drawn through *n* parallel to R in the point *n''*, and then supposes the effect of R to be distributed over the area *mn''* only of the joint *mn'*. The following is a general statement of his argument:—

Fig. 2.



"It is evident," he says, "that the distribution of pressure over joint *nm* is greater than that over a joint such as *mk*, situated above it, because in the latter case the inclination of R is more acute with regard to the joint's surface, and the pressures are diminished by those due to the

weight of masonry kmn, and to the fluid pressure acting over surface nk, both of which affect the intensity of pressure at m." Such, however, he argues, is not the case with joint mn'', taken below mn, because the weight of masonry mnn' can have no effect, as likewise the fluid pressure over nn' on the pressure at m, and there is no reason why the triangle nmn' should not be regarded as *simply transmitting* the pressure it receives from above. Under these conditions, joint mn', which is perpendicular to the direction of the resultant pressure R, would become the joint yielding the maximum intensity of pressure at m, and such being the case, it will be seen from *Fig.* 2, that—

If $nm = l$, $om = u'$, $mn'' = l'$, $o'm = u''$, and α be the angle of inclination of R to the vertical; and if $mo = \frac{l}{3}$, then will $mo' = \frac{l'}{3} = \frac{l \cos \alpha}{3}$, and $u'' = u' \cos \alpha$ and, we shall have for p, the maximum intensity of pressure at m—

$$p = \frac{2\,\mathrm{R}}{l \cos \alpha}\left(2 - \frac{3\,u'}{l}\right)$$

instead of the value usually given, *viz.*, $\frac{2\,\mathrm{R} \cos \alpha}{l}\left(2 - \frac{3\,u'}{l}\right)$, which is a very serious difference, but altogether on the side of safety.

The objections offered to this argument are, that it is inadmissible to take one property of matter into consideration, (*e.g.*, its property of transmitting pressure) and neglect another (its property of being attracted earthwards) as likewise to regard the resultant pressure R as distributed over only a portion of joint mn' and not the whole (in other words alter the dimensions and inclination of the joint's surface relatively to the force by projecting its area by rays parallel to the direction of the force and not at the same time suppose the magnitude and direction of the force correspondingly altered relatively to the joint) because the weight of the masonry mnn', as likewise the fluid pressure over the area nn' *do* affect the intensity of pressure at m, causing an increase in the magnitude of the resultant pressure over joint mn', and an alteration in the position of its centre, which will certainly not be at o', but move further towards n', being the centre due to the weight of the wall Amn'B combined with the fluid pressure over the surface Cn'.

* Annales des Ponts et Chaussées, 5th Series, Vol. X., p. 173.

APPENDIX D.

ON WELL-FOUNDATIONS.

By Lieut.-Col. Allan Cunningham, R.E., *Hon. Fellow of King's College, London.* (*Reprinted from Paper No. LXXXIII.,* "*Professional Papers on Indian Engineering,*" *Vol. II., Second Series*).

CONTENTS.

1. *Objects proposed.*—It is proposed to investigate in this Paper the Problems of STABILITY and STRENGTH OF *Masonry Well-foundations in quicksand.*

These are Problems of great practical importance, as the Piers of most of the Railway Bridges over the great Indian rivers are in general simply large Masonry Wells sunk in many instances in quicksand beds.

[This Paper is intended to be a complete investigation of the whole Problem, as far as the present state of science admits, with the final results in a form immediately suitable for calculations of the practical Engineer: all "results" are accordingly either reduced to simple algebraic formulæ or to "simple statements of fact," and numbered *seriatim.* All detailed mathematical investigations are separated into an Appendix.

A complete numerical solution of one actual Example is given in Arts. 57, 69, with full references (for all formulæ used) to the Text, so that no difficulty should occur to the practical Engineer in applying the principles and formulæ of this Paper to practice].

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2. It may be premised that the particular practical case which gave rise to this investigation was that of the Well-foundations of certain Bridges over the Ganges, Rámgangá, Sai, Gumti and Garrá rivers on the Oudh and Rohilkhand Railway.

The Bridge over the Ramganga has been selected as an example in illustration of the Methods and Formulæ of this Paper, as being one of the most unfavorable cases, the soil being simply quicksand.

An outline diagram of one of the Piers is given (*Fig.* 1, *Plate* A). It will be seen that the Pier is sunk 75 feet below the cold weather bed of the stream, and it is supposed that in the worst floods scour might take place to a depth of 50 feet. The greatest surface velocity is believed not to exceed 16 feet per second.

The Girders of the Bridge expose a large surface (317 square feet from Pier to Pier) to the Wind The Piers themselves are liable to the shock of Drift Timber Logs, and Rafts, and of laden Boats. The numerical data are taken from the official Railway Records.

Summary of Results.

3. In consequence of the length of this Paper, it is considered advisable to give a general summary of the Results.

Section.	Art.	Result.	Brief statement of Results.
SECTION I.—INTRODUCTION.	5		A "slender Pier in a rapid current over a quicksand bed" may fail in four ways, *viz.*, by *sinking*, *sliding*, *tilting*, or *snapping*: these give rise to four distinct problems.
	7		The data for these problems are very imperfect.

Section.	Art.	Result.	Brief statement of Results.
SECTION I.—INTRODUCTION, (*continued.*)	7		A proper numerical solution at present impossible. Useful generalizations may however be certainly drawn, and a numerical solution may be found on certain hypotheses. These are the objects of this Paper.
	9		The Problem treated as a Statical Problem. Impact can only be imperfectly allowed for.
	11		The External Forces divisible into two sets of parallel forces, *viz.*, Vertical, and Horizontal (parallel to stream).
SECTION II.—VERTICAL FORCES.	13	4°	The Total Friction developable from the subsoil is *alone sufficient* to support the Weight of the Well.
		6° 7°	Quicksand can at great depths bear very great *direct* pressure, and also yield great *tangential* resistance, and is therefore semi-solid.
	15		Water-percolation does not affect the "Whole Pressure" of the upper courses of Masonry on the lower: that "Whole Pressure" is *always* simply the Weight of those courses.
	16		Water having access to a Well's base exerts an *upward* pressure equal to the Weight of the fluid displaced, which is wholly conducive to Instability (of Rotation). In questions of Vertical and Rotary STABILITY this is conveniently allowed for by reducing the "effective heaviness" of the immersed masonry by $62\frac{1}{2}$ lbs. per cubic foot.
	18	2°	Water-percolation seriously reduces a Well's TRANSVERSE STRENGTH. The Masonry of the Wells should therefore be set in good cement, and bonded vertically with iron ties.
	20 21	(3) (6)	The Vertical subsoil Re-actions, both "Direct" and Frictional, always set *upwards*, and are always conducive to INSTABILITY.
	22		There appears danger of Wells failing *by sinking* under the peculiar action of the Disturbing Forces after full scour has taken place. Experimental data not available for a numerical estimate.
SECTION III.—HORIZONTAL FORCES.	23		Records of Wind-pressure for India very imperfect. Maximum Wind-pressure in Oudh is 40 lbs per sq. ft, and in Lower Bengal 50 lbs. per sq. ft.
	24 26	(7) to (12)	Formulæ for Total Wind-pressure, and Moment of the same.
	27 34		Laws of Current-pressure complex, and imperfectly known.

Section.	Art.	Result.	Brief statement of Results.
SECTION III.—HORIZONTAL FORCES, (*continued*).	28		Subsurface velocity varies as abscissæ of a parabola whose ordinates represent depths.
	29 to 39	13 to 29	New and simple formulæ proposed for Subsurface Velocity, and for Intensity, Total, Centre, and Moment of Current-pressure on Well and on Drift Mass.
	41	34	The Disturbing Forces increase both the up-stream and down-stream Horizontal Re-actions of the soil, and raise the centre of pressure of the former and depress that of the latter.
	43		Wells should be sunk below level of "no motion" of sub-soil. This level is in this Problem the "virtual bed" of the current.
SECTION IV.—ROTATION.	45		Problem of Rotary Stability is complex, and resembles that of Stability of ships.
	46		Point chosen as "Centre of Moments" indifferent. The Re-actions of soil cannot be disregarded (as in Structures on rigid Foundations).
	48	(38)	Resultant Moment of *all* the Vertical Forces is a Moment of Stability.
	49	(39)	Increased sinking increases the Moment (of Instability) of the Disturbing Forces, and also the available Resultant Moment (of Stability) of Horizontal subsoil Re-actions, the latter in a *higher* ratio.
	ib	(40) (41)	The Horizontal subsoil Resistance is the *chief* element of Stability of Rotation, and the Pier's weight is a comparatively unimportant element.
	ib.	(41)	Stability of rotation can only be secured by deep sinking.
	50		Stability of rotation cannot be *with certainty* numerically estimated for want of experimental data on nature of subsoil re-action.
	51		By making certain hypotheses a *highly probable* numerical value may be found for the intensities of pressure on the soil caused by the Disturbing Forces.
	54 55	(49) (53)	Formulæ for Total of, also for Mean and Maximum Intensities of, Horizontal subsoil Resistance on these hypotheses.
	55		It rests with the practical Engineer to decide in each case whether a particular soil can bear these pressures.
	57		Example—Pier for Rámgangá Bridge. Maximum pressure-intensities on subsoil found.

Section.	Art.	Result.	Brief statement of Results.
SECTION V.—TRANSVERSE STRENGTH.	61		The Plane of Greatest Stress lies at a short distance below the current bed, probably about $\frac{1}{8}$ of depth of sinking. The Stresses at this plane only require investigation.
	62		No reliance should be placed on vertical Tenacity of Masonry simply set in mortar.
	65	(54) (54*a*)	Formulæ for Greatest, Mean, and Least Stress-intensities at plane of Greatest Stress
	68		Slender Piers suffer *some* Tensile Stress *nearly throughout* their length. Such Piers if simply set in Mortar are dangerously liable to fail by opening of joints under Tension; they should for safety be tied with vertical Iron Ties throughout.
	68		A solid cylindric pier is one of the *weakest* forms as regards Transverse Strength.
	69		Example.—Pier for Rámgangá Bridge. Longitudinal Stress intensities found. This Pier is strong enough if tied with vertical iron ties throughout, but if *simply set in mortar* its Transverse Strength is doubtful.

4. *Practical Conditions.*—The great rapidity of the large Indian rivers in flood, and the shifting nature of their beds (often quicksand) lead to the practical condition that—

1°. "The natural waterway must be as little as possible diminished by introduction of Piers," and consequently

2°. "The Piers must be as slender as is compatible with the requisite STABILITY and STRENGTH."

5. STABILITY *and* STRENGTH.—The consequence of these particular practical conditions, *viz.*, "a slender Pier in a rapid current over a quicksand bed" is that the *complete* Treatment of the Problem requires the consideration of a number of elements most unusual in Masonry Structures, *viz.*, of the *distinct* problems of STABILITY and STRENGTH, and moreover of several distinct forms of the former. In fact the Wells appear liable to fail in four distinct ways, specified below, each of which gives rise to a distinct Problem as stated.

No.	Manner of failing.	Problem of
1	By sinking as a whole,	Vertical Stability.
2	By sliding as a whole,	Lateral Stability.
3	By tilting over as a whole,	Rotary Stability.
4	By cross-breaking or snapping,	Transverse Strength.

[Liability of failure by Shearing is omitted from the above enumeration, as it is matter of practical experience that *solid* Masonry structures do not fail by shearing across under Transverse Strain].

It appears that Well-foundations as hitherto constructed have usually failed by tilting over as a whole, *i.e.*, by want of Stability of Rotation. This Problem will therefore receive especial attention, *see* Section iv.

6. *General interest of the Problems.*—It is very seldom that so many distinct manners of failure really require consideration in Masonry Structures. The present Problems present in consequence considerable interest even from a theoretical point of view.

N.B.—The Problem as set forth in its generality is *an almost untried* one. The author has had the advantage of consulting a Report on a portion of the general Problem, made for information of the Oudh and Rohilkhand Railway, by E. Bell, Esq., C.E. Mr. Bell's Report deals *solely* with the Problem of TRANSVERSE STRENGTH. Mr. Bell considered the Problem of STABILITY at present *insoluble*.

The author has had the advantage of discussing the general Problem of STABILITY with Mr. J. Elliott, M.A., Mathematical Professor, Muir College, Allahabad, and is much indebted to him for advice. The general treatment of the difficult question of Friction (Art. 21) is due to Mr. Elliott.

7. *Imperfection of the data.*—The chief difficulty attending this problem is the great imperfection of the practical data, and particularly of the two following:—

(1). Laws of Current pressure.

(2). Laws of Resistance of subsoil.

The laws of the former are imperfectly known, but so little is known of those of the latter, that, as will appear in the sequel, it is simply impossible, at present, to produce with real certainty any definite numerical solution of any of the four Problems proposed in Art. 5.

Nevertheless, as will appear, many useful generalizations may be certainly drawn from a critical discussion; further, by adopting certain probable hypotheses as to the nature of subsoil Resistance, definite numerical solution of the Problems may be obtained.

This Paper aims therefore—

(1). At establishing useful generalizations.

(2). At definite numerical solutions of the Problems upon certain probable hypotheses as to nature of subsoil resistance.

External Forces.

8. The External Forces may be divided into two sets: (1), APPLIED FORCES; (2), RE-ACTIONS, which are of course developed by the former.

The Applied Forces are of two kinds—

1°. VERTICAL: these are simply the Weight of the Superstructure and Structure.

2°. HORIZONTAL or nearly so: these are

(1). Wind on the Superstructure.
(2). Wind on the Piers.
(3). Impact of floating Drift
(4). Current on floating Drift caught by the Piers.
(5). Current on the Piers.

It is convenient, for brevity, to class the whole of these last under the general term "Disturbing Forces."

The Re-actions will be considered hereafter.

General Treatment.

6. The first question that naturally presents itself is the following:—

"Is the Problem to be treated as a question in Dynamics (Kinetics), or in Statics"?

Inasmuch as the "Disturbing Forces" (Wind, Current, and Impact of Drift) are all *vires vivæ*, the proper scientific treatment would be as a problem in Kinetics. The question would thus present itself in this form:—

"Is the Potential Energy of the Re-action equal to the Kinetic Energy of the 'Disturbing Forces'?"

The estimation of Energy would require that *all* the co-efficients of elasticity of the masonry and subsoil should be known. As these co-efficients are however quite unknown, the problem cannot, at present, be solved in this form.

Two of the Disturbing Forces, *viz.*, Wind and Current, have however been reduced, by experiment, to equivalent STATICAL PRESSURES, so that as far as these are concerned, the Problem may be treated as a question in Statics, *i.e.*, one of equilibrium.

No such experimental reduction has however been made in the case of Impact of solid bodies, *e.g.*, Drift-masses, so that the effect of Impact cannot *really* be included in a solution as a case of Statics, and for the reasons before given the Kinetic solution is also impossible.

An imperfect equivalent for the *effect* of Impact, but one which will probably meet all practical requirements, is to substitute a "Mass of floating débris" as caught against the Well and there exposed to the full power of the current, so that "Impact" is *replaced* by a STATICAL PRESSURE.

N.B.—The size and shape of this hypothetical floating mass must be assigned as a *purely empirical* question by the practical Engineer. Theory affords absolutely no guide to this assignment. A mass presenting 100 square feet broadside to the current has been assumed in the Example, Arts. 57 and 69

The whole of the Disturbing Forces having been thus reduced to Statical Pressures, the Problem may now be treated as one in Statics.

10. It will be assumed that—

"Piers should be designed to meet the case when the Disturbing Forces *simultaneously* attain their maxima, and also set in the *same* direction (down-stream)."

[This is obviously the condition most unfavorable to the Piers, for under any other condition the Disturbing Forces will be either—(1) actually not at their maxima or (2) partially counteracting each other].

11. *Resolution of the External Forces.*—The Vertical Forces are necessarily "a parallel system." The Disturbing Forces may be assumed, with sufficient accuracy for this Problem, to be *horizontal* in direction, and by the limitation to the case of their setting in the *same* direction (Art. 10) they become "a system of parallel horizontal forces." Moreover in symmetrical Wells, the usual case, the Disturbing Forces may be assumed, with sufficient accuracy for this problem, to be *symmetrically distributed* about the vertical axis of the well. It follows that,—

1°. The resolved parts, across the stream's direction, of the action of the Disturbing Forces on the curved outline of the Well balance each other.

2°. The Resultants of each of the Disturbing Forces pass through that axis.

Inasmuch as the Re-actions must be equal and opposite to the Applied Forces, they may be similarly classified.

It follows that the "External Force" may, for most purposes, be reduced to two sets of "parallel forces," *viz.*,

1°. A set of vertical forces.

2°. A set of horizontal forces, parallel to the stream.

It is convenient to consider the Vertical and Horizontal Forces separately, and in the order indicated in the Table of Contents, *q. v.*

Section II.—Vertical Forces.

12. *Vertical Forces Classed.*—The question of the "effective heaviness" of *immersed* masonry is really a very serious one, as enormously affecting the *quantity* of Masonry requisite to Stability. As this is a question on which the most opposite opinions have been advanced,

it will be discussed at some length. Consider all the vertical forces: these are*

1°. The WEIGHT of the pier, ($-$W).

2°. The WATER-PRESSURE, *upwards* or *downwards* on all parts to which fluid has access *from below* or *from above* respectively, (W′).

3°. The RE-ACTION of the soil, upwards, (R).

4°. The FRICTION (F) between the sides of the Pier and the soil, *upwards* or *downwards*, according as the tendency of the Pier is at any moment to *sink deeper* or to *rise* vertically—or the vertically resolved parts of the partial Frictions on all sides of the Pier, when the tendency to motion is not vertical.

Then, clearly *at all times* when the Pier is neither sinking nor rising, the following equation obtains

$$-W + R + W' \pm F = 0 \qquad \text{......................(1).}$$

13. *Experimental evidence.*—The chief difficulty attending this part of the investigation is the *want of experimental evidence* on the nature of "internal subsoil pressure." The following are apparently the only known data: as experimental evidence is the basis of all scientific investigations, they will be made the basis of the present investigation. The inferences which will be drawn from them should be carefully considered, as most of this investigation is simply the necessary conclusion from those "inferences" as premisses.

It appears that—

(1). In some Wells the water has been known to stand at a higher level (during the period of sinking) inside the Well than outside it.

Inference. It follows that in those cases—

1°. Quicksand, even at great depths, permits thorough permeation of water, so that water has *access to the base* of Wells in quicksand.

2°. The "internal fluid pressure" at the base of the Wells was *somewhat greater* than the "hydrostatic pressure" (of the water).

Also in some Wells—

(2). It has been found, during the period of sinking, that the weight of the (then hollow) Well was not sufficient to cause its own sinking, even when all direct support from the subjacent soil had been removed, by the removal of that soil; and further, that the Weight of the hollow Well together with the heaviest Weight of many tons of iron rails that can in practice be laid on the Well is, after sinking a great depth, not sufficient to make the Well sink further, even when all direct support from the subjacent soil has been removed, by removal of that soil; in fact, that all practically available mechanical appliances eventually fail to produce further sinking.

Inference.—Since by the removal of the subjacent soil, its *direct* Re-ac-

* It is convenient to estimate *upward* Forces as positive: all Weights being downward Forces are thus negative.

tion ceases, *i.e.*, $R = 0$, hence $-W + W' + F = 0$, in such cases: thus it appears that—

3°. The Total Weight (W) of the Well and its superstructure were supported solely by the upward Water-pressure (W′) and Friction (F).

4°. The Total Vertical Friction (F) which can thus be developed in support of the Pier is *very great.*

5°. The Total Normal Pressure of the subsoil against the sides of the Pier, by which cause alone can the Friction be developed, must be *very great indeed.*

6°. Quicksand is able, at any rate at great depths, to sustain very great *direct* normal pressure.

7°. Quicksand, although thoroughly permeable by water, even at great depths, (*see* Inference 1°), is at great depths capable of exerting considerable "Tangential Resistance," to which property alone the Friction is due, and is therefore at such depths a "semi-solid," or *very imperfect* fluid.

N.B.—Sensibly perfect still fluids (*e.g.*, still water) exercise no sensible Tangential Resistance.

The great practical importance of some of these inferences, especially 1°, 4°, 6°, 7°, will appear in the sequel.

14. *Internal fluid-pressure.*—The law of variation of "internal pressure" in *current fluid* is involved in equations which have not yet been solved. It is however known in a general manner that pressure *decreases* with increased velocity, so that "internal fluid-pressure" in current fluid is generally *less* than the *hydrostatic* pressure.

It appears nevertheless (*see* Inference 2°) that the "internal fluid-pressure" at the base of some of the Wells has been known to be somewhat *greater* than the hydrostatic pressure of the water: this is probably due to the quicksand being a sort of "imperfect fluid," of greater density than water.

It appears, therefore, that it will be a sufficiently approximate method for the present problem to estimate the "internal subsoil fluid pressure," (*i.e.*, of the quicksand or mixture of sand and water) as *the same* as the "hydrostatic pressure" of the water alone.

[At the same time that the difficulty of estimating "internal fluid-pressure" of current fluid is thus avoided, it is evident that *some* allowance has been made for semi-fluidity of quicksand: as to the sufficiency of this allowance, be it observed that the allowance required can hardly be very great, as it seems difficult to imagine that the subsoil could supply the great Total Friction known to be developable (Inference 4°) unless its state of aggregation at great depths was more approaching to that of a solid than a fluid (*see* Inference 7°).

On this particular hypothesis, no further notice need be taken of the "semi-fluidity" of the quicksand].

15. *Water-pressure.*—Still water is, unless absolutely confined, capable of yielding and also of transmitting only its proper "hydrostatic pressure," neither more nor less than that due to its depth, and that only *uniformly* and in all directions at once, and cannot therefore transmit either variable Pressure, or *any* Tension.

Further there is, necessarily, an *upward* Water-pressure on all parts to which fluid has access *from underneath*, which might be supposed to diminish the "effective weight" of the masonry above, but as this is necessarily accompanied by a simultaneous and equal *downward* Water-pressure on the parts below, to which the fluid of course has access *from above*, it follows as a resultant effect that the—

"Whole Pressure of each horizontal Stratum on those below is simply *its own Weight*,"—or in other words

"Permeation by water does not alter the TOTAL PRESSURE of the upper courses of Masonry on the lower," although, as presently shown, it may greatly alter the *distribution* of that Pressure.

16. *Effect of Water-pressure on Stability.*—As regards STABILITY, both Vertical and Rotary, of the Pier as a whole, also as regards STABILITY, of both kinds, of the part above a fracture (should complete fracture occur below water level), the *upward* Water-pressure exerts an *upward* "Re-action" against the base and parts to which it has access from below, which is equivalent to reducing the "effective heaviness" of the immersed masonry by about 62·5 lbs. per cubic foot (the "heaviness" of water).

[*N.B.*—This may be conveniently allowed for, in questions of Vertical and Rotary Stability only, by estimating the "effective heaviness" of immersed masonry as 62½ lbs. per cubic foot less than that of masonry in air. It must be borne in mind however that this method is suggested solely for *convenience* in numerical calculation. It must be carefully remembered that the "whole weight" of masonry, and also the "whole upward Water-pressure," are *distinct* forces, the former wholly effective in producing STABILITY, the latter wholly effective in producing INSTABILITY].

17. *Effect of Water-pressure on Transverse Strength.*

In any case, the Water having access to the base of the Pier (*see* Inference under 1°) yields an *upward* Pressure (*viz.*, its "hydrostatic pressure") *uniformly distributed* over the base. This affects Transverse Strength to the extent that the distribution of Stress throughout the Pier must be such that the Pier shall yield a *downward* Pressure on its base, which must *not be less at any point* of its base than that "hydrostatic pressure", (but may be greater).

18. The Transverse Strength is further affected as follows:—

1°. If the Pier be impervious to water.

2°. If the Pier be in parts permeated by water.

1°. *Pier impervious.*—If the Pier be impervious to water, the state of Longitudinal Stress throughout the Pier is only affected as above stated.

2°. *Pier pervious.*—If the Pier be partly permeated by water: then inasmuch as a *solid* Pier under Transverse Strain (due to the applied Forces of Wind and Current) is at any horizontal plane "in a state of Longitudinal Stress" which is generally supposed to be uniformly-varying with the distance from a certain 'neutral axis' in that plane; it follows, from the nature of fluid-pressure, *v. supra*, that, should water have access to any portion of such a plane, then, when equilibrium is established, the *distribution* of the Longitudinal Stress is *completely altered over that plane*, there being substituted *over the wetted portion* the *uniform* "hydrostatic pressure," so that the algebraic difference between the previous Stress, on that portion, and the hydrostatic pressure must now be borne by the *remaining solid* material at that plane.

It is easy to see in a general manner that the effect is usually unfavorable to Transverse Strength, especially if the Transverse Strain be so great as to cause actual Tension on one side of the Pier, as the permeation of that side would greatly increase the Tension on the remaining solid material (because the fluid is unable to transmit *any* Tensile Stress): unfortunately this is the very Stress which the Material (Masonry set in mortar) is least fit to bear, and the side in Tension is at the same time the most liable to percolation.

Practical Remark.—It is therefore very desirable that the Masonry "should be set in good *cement*, and bound with vertical, longitudinal, iron ties, so that water percolation may be prevented both during the period of sinking, and as a permanency."

[It will be assumed throughout this Investigation that this point has been attended to, and that the effect of Water-percolation on Transverse Strength (which would be very complex in detail) need not be considered].

19. *Vertical Re-actions of the soil.*—These are of two kinds—

I. Direct—exerting direct upward pressure on the base.

II. Tangential—being the vertical portion of Friction on the sides.

It seems impossible to separate the effects of these two (none of the experimental date being available): the consideration of these effects presents considerable difficulty, when the Pier is under the action of the Disturbing Forces, *except as a statement of general principles.*

A Pier *not under lateral applied Forces* simply tends to sink vertically, so that the Vertical Re-actions are distributed, and, for the case of a symmetrical solid Pier (the usual case in practice), are distributed *uniformly* (1) the Direct Re-action (R) over the base, and (2) the Tangential Re-action (Friction, or F) around the Pier, and set *upwards*, their sum being

$$R + F = W - W'.$$

It seems sufficiently evident that the only effect of the Disturbing Forces, being horizontal, on the vertical Re-actions is—so long as equilibrium holds—to alter their *distribution* and therefore to alter the *position* of their Resultant, but not to alter the *magnitude* of that Resultant which remains at the same amount, *viz.*, $(W' + R + F) = W$, otherwise the Pier would sink or rise, as a whole, vertically, which would destroy its use as a support for Girders.

Observing now that the Upward Water-pressure (W') is for a *very slight* displacement (tilting) of the Pier a quantity sensibly constant, and that the Weight of the Pier (W) is also a constant, it follows that—

"The Total Vertical Re-action of the subsoil $(R + F)$ is constant for a very slight displacement (tilting) of the Pier,"

i.e., $(R + F) = W - W'$ a constant quantity,..........(2).

But it seems almost impossible to *separate* these forces R, F : it is known, *see* Art. 11—(2), that the Total Vertical Friction (F) developable is in the absence of the Direct Re-action $(R = o)$ *of itself* sufficient to support the Weight of the hollow Pier together with the heaviest Load that could in practice be laid on it—when not under the action of the Disturbing Forces. Nothing else is *certainly* known.

20. *Direct Vertical Re-action of the soil.*—It was shown, *see* Inference (5), that quicksand is at great depths capable of sustaining great pressure and therefore of yielding great *direct* Re-action (R).

[*N.B.*—Although in the course of sinking, the subjacent soil is continually removed, so that the Pier does not *then* rest *on* the soil, still it seems almost certain that the contact of the soil with the base of the Pier is eventually renewed either by the final hearting of the Pier with concrete, or by the imperfectly fluid quicksand eventually refilling any vacancy that may be left below the Masonry].

It has been explained (Inference 7°) that quicksand is at great depths probably a sort of very imperfect fluid; reasons were given (Art. 14) for assuming that a sufficient allowance had there been made for the "fluid-resistance" of the quicksand, and for assuming it was at great depths otherwise a semi-solid capable of sustaining considerable tangential stress, and therefore also capable of bearing direct pressure *of varying intensity*, and therefore also of yielding Direct Re-action of like varying intensity.

The general effect of Lateral Forces applied to a Pier resting on such a material as supposed, would be to alter the distribution of the pressure of its Weight on that material, *viz.*, to diminish the pressure on the *near* side and increase it on the *far* side, thereby throwing the Resultant Pressure

towards the far side; the direct Re-action of the soil would of course exactly follow suit: and this effect would go on increasing with an increase of the Applied Forces to an extent depending on the actual power of the subsoil of sustaining varying pressure.

But inasmuch as it seems impossible to suppose that the quicksand can be in a state in any way approaching sensible rigidity (as for instance of rock foundations), it follows that there is a limit to the extent to which the variation of pressure-intensity can proceed, and that the Resultant of the Direct Resistance of the soil on the base can never approach very near the far edge, with which it would eventually sensibly coincide in a very firm foundation, but must always fall *considerably within* the base.

Hence, observing that the Re-action is an *upward* Force, the following important result:

"The Direct upward Re-action of the soil is always *conducive to Instability*" (of rotation),.......... (3).

21. *Tangential Vertical Re-action of the soil.*—This is the vertical portion of all partial Frictions around the pier.

[*N.B.*—This Re-action is in the present problem a very important one, as the experimental evidence (*see* Art. 13) is to the effect that it *alone* may support the Weight of the Pier together with the heaviest Load that can in practice be laid on it.

It is known that Friction is a *Tangential* Force between particles of material that are in mutual contact, and always opposite to the direction of incipient relative motion.

In a symmetrical solid Pier not under the action of Lateral Forces the partial Frictions are (as already observed, Art. 19) *uniformly distributed around* the Pier and set *upwards*.

But can they ever set *downwards?* It has been suggested* that when under the action of Lateral Forces, they can and do set *downwards*, and even with such intensity as to *neutralize* the *upward* Water-Pressure. If true, Friction would be a *most important* element in producing STABILITY (of rotation). It seems therefore advisable to discuss this question somewhat fully].

The first effect of applied Lateral Forces will be to diminish the pressure of the soil on the near side, and increase it on the far side, thereby diminishing the vertical friction-intensities on the *near* side, and increasing them on the far side (inasmuch as the friction-intensities are proportional to the pressure-intensities); and this effect will go on increasing as the lateral applied Forces increase until the instant of incipient motion. At that instant every particle of masonry on the *near* side is about to *rise*, but in such a manner that it tends to move *altogether away from* all the imme-

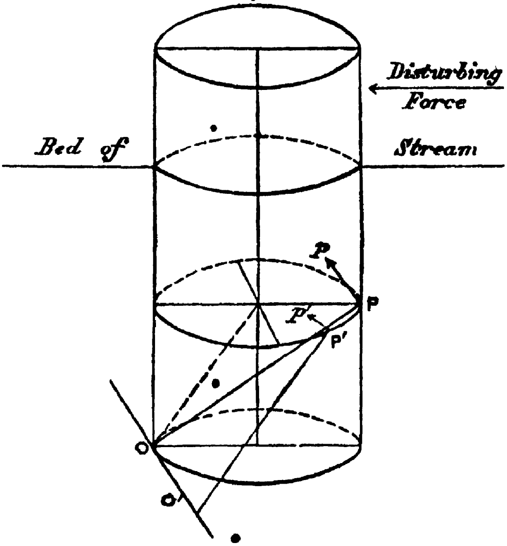
* By E. Byrne, Esq., Oudh and Rohilkhand Railway, Lucknow, in "Calculations of Stability" of certain bridges, 27th May, 1872.

diately contiguous particles of soil, and not *tangentially* along them (in which case only could Friction exist), so that the following important result follows:—

"All friction (in *every* direction) *ceases* all over the *near* side at the instant of incipient motion,".. (4).

This will be evident from *Fig.* 8; the particles of masonry P, P′, clearly tend to move in the direction Pp, P′p′ (at right angles to OP, O′P′ which are perpendiculars from P, P′ on the line OO′, the *axis* of rotation, so that OP, O′P′ are radii of, and O, O′ centres of, motion of the points P, P′), *i.e., away from all contiguous particles of soil, and not to slide over them.*

Fig. 8.



Kinetic friction.—It seems probable that, at the same time that the Pier tends to move (as a whole) away from its contact with the soil on the near side (*i.e.*, side of application of the lateral Forces), the soil being quicksand tends to fill the vacancy, and so renew the contact; the particles of sand in so moving probably *fall, i.e.*, impinge in a somewhat *downward* direction against the masonry, thereby expending part of their *downward* "vis viva" (Energy of motion) on the mass of the Pier, and also causing *downward* "friction of motion," which is of course conducive to Stability.

The *nascent* velocity of the impinging sand-particles will however be *very small*, so that their nascent downward Impact and the simultaneous nascent downward Kinetic Friction will also be very small, and should be neglected in comparison with the large Forces in action.

The state of the Frictional Forces over the far side is more difficult to form a clear conception of. In the state of incipient motion (round the point O in the far edge of the base), every particle of masonry on the *far* side tends to *rise* and at the same time *press harder* on the contiguous soil, so that *true sliding of the masonry over the sand tends to take place:* true Friction is thus developed,—opposite at every point to the direction of incipient motion of that point, and proportional to the pressure at that point; and therefore—*generally different both in direction and intensity at*

every point of *the far side.* Resolving these partial Frictions vertically, the following important result follows:—

"At the instant of incipient motion, the partial vertical Frictions are zero at the diametral plane which lies across the stream (*i e.*, perpendicular to the direction of the applied lateral Forces), and increase in intensity towards—and attain a maximum at—the far vertical edge of the Pier," .... (5).

The actual distribution of the vertical friction-intensities, and the actual intensity at any one point, are entirely unknown. There is a natural limit to the maximum friction-intensity, *viz.*, that due to the maximum pressure-intensity which the soil can bear.

It is possible that,—at the same time that the distribution of vertical Friction changes as explained,—some of the partial vertical frictions may set *downwards;* it is possible also that their Resultant (F) may change in magnitude; but, whatever that change may be, the equation

$$R + F = W - W' \text{ (a constant quantity)},$$

must certainly *obtain at all times*, otherwise the Pier must sink, *i.e.*, *the Resultant Vertical Re-action of all kinds* (R + F) *must be unchanged both in magnitude and direction.*

It is obvious from what precedes that the Resultant (F) can never approach to coincidence with the far edge of the Pier, but must always fall *within* the base.

Observing also, that it is hardly possible that the Direct vertical Re-action (R) of the soil (quicksand) could of itself sustain the whole downward pressure (W − W′) there follows the important result—

"The Resultant Vertical Friction always sets *upwards*, and is always conducive to Instability," .... (6).

Vertical Stability—Stability of Sinking.

22. It is clearly essential to the use of a Pier that it should not sink further, once the superstructure is commenced.

As there seems (to the author) danger of Piers in quicksand sinking further under the peculiar action of the applied Forces, the question should receive consideration.

Assuming that the Wells are sunk till mechanical appliances fail to sink them further, and also that the superstructure is put on *before* the maximum scour has taken place, (an almost certain event, as the supposed maximum scour might not occur for years), it is known that the *actual*

depth of soil sunk through can yield sufficient vertical Resistance (Re-action) of all kinds (*i.e.*, both direct upward Pressure on the base, and upward Friction on the sides), to support the greatest Load (say W) that has ever been placed on the Pier, so that *at that time* (previous to maximum scour) the equation $-W + W' + R + F = 0$, certainly obtains.

But when full scour has taken place, *many feet* (sometimes 50 feet) in depth of the soil which previously supplied the Total vertical Friction (F), are completely removed, and the remaining subsoil, still in contact with the Pier, is probably much reduced in cohesiveness—both by the removal of the superincumbent sand, and by more thorough water-percolation—and therefore in power of yielding direct pressure, whether on the base or on the sides (by which latter pressure alone can friction be developed).

But the action of the "Disturbing Forces" is further unfavourable to STABILITY of this kind, because they tend to alter the distribution of the partial vertical Re-actions, both Direct and Frictional, which the soil must supply, by accumulating the greatest intensity of those Re-actions towards the far side, so that, though the soil is called on to supply only the *same* Total Vertical Re-action ($R + F = W - W'$) as before, it has to supply it in a far less favourable manner, *viz.*, by supplying pressure, upwards on the base, normal on the sides, of far greater intensity in the neighbourhood of the far (down-stream) side than before. There is of course a natural limit to this, *viz.*, the greatest pressure-intensity which the soil can supply, after full scour has taken place. If the partial pressures caused by the external Forces anywhere exceed this, motion must take place, *i.e.*, the Pier *must sink*.

[The experimental data required for properly estimating numerically the greatest pressure-intensity that the Disturbing Forces will cause are entirely wanting. An attempt will be made on certain hypotheses later].

SECTION III.—HORIZONTAL FORCES.

WIND.

23. *Wind-intensity.*—The data for *maximum* intensity in India are very defective; it may be fairly assumed that the Wind will *occasionally* reach a Maximum whose "Statical equivalent" is 50 lbs. per square foot in Lower Bengal; 40 lbs. per square foot in Oudh.

24. *Total Wind-Pressure.*—It is considered sufficiently accurate for

the purpose in view to estimate Wind-pressure as *horizontal* and of *uniform intensity at all moderate heights*, also to estimate as follows* for effect on *curved* Piers :—

$$\left.\begin{array}{l}\text{Total Wind-pressure (parallel to}\\ \text{Wind's direction) on a Vertical}\\ \text{cylinder or conic frustum, ..}\end{array}\right\} = \tfrac{1}{2} \text{ of } \left\{\begin{array}{l}\text{Total Pressure on a Ver-}\\ \text{tical diametral plane}\\ \text{broadside to wind, ...}\end{array}\right\} \text{......(7).}$$

Then if w' = Maximum Wind-pressure in pounds per square foot.

A_G = Area of Girder-surface exposed to Wind in square feet.

A_M = Area of Vertical Diametral section of Pier exposed.

P_G = Total Wind-pressure on Girders.

P_M = Total Wind-pressure (parallel to Wind) on Masonry exposed.

$$P_G = w'.\ A_G \text{ ....(8).}$$

$$P_M = \tfrac{1}{2}\ .\ w'\ A_M \text{ ....(9).}$$

25. *Centres of Wind-pressure.*—It follows from the preceding that—

1°. The Centre of Wind-pressure on Girders is near the middle of their height, when the Girders have, as would usually be the case, straight flanges, and have tolerably equal areas of metal exposed to the Wind on either side of their mean line ("neutral surface").

2°. The Centre of Wind-pressure on the Piers is the centre of gravity of the exposed vertical diametral section. But inasmuch as in practice the taper of the Piers is usually very slight, this point is near the middle of the height of that exposed section.

It is considered that it will be sufficiently accurate for the present Problem to assume that—

"The Centres of Wind-pressure on the Girders and Piers are at the middle of their heights," .... (10).

26. *Moment of Wind-pressure.*—It follows from the preceding that this may be expressed in the following simple manner :—

$$1°.\quad \left.\begin{array}{r}\text{Moment of wind-pres-}\\ \text{sure on Girders, ...}\end{array}\right\} = \left\{\begin{array}{l}\text{Total Pressure} \times \text{height of mid-}\\ \text{dle of Girders above centre of}\\ \text{moments, ...}\end{array}\right\} \text{..... (11).}$$

$$2°.\quad \left.\begin{array}{r}\text{Moment of Wind-pres-}\\ \text{sure on Piers, ...}\end{array}\right\} = \left\{\begin{array}{l}\text{Total Pressure} \times \text{height of mid-}\\ \text{dle of exposed area above}\\ \text{centre of moments, ...}\end{array}\right\} \text{...... (12).}$$

Current-pressure and its Moment.

27. It is known that Current-pressure varies as square of velocity which is *itself variable* with the depth.

It follows therefore that neither Total Current-pressure nor its Moment can be expressed by any such simple means as in the case of Wind-pressure.

* Rankine's Applied Mechanics, Art. 215.

It will be necessary first to investigate formulæ for the sub-surface velocity, and thence for pressure-intensity, Total Pressure, and Moment.

[The formulæ about to be given have been constructed by the author for this Problem: their detailed construction is given in the Appendix to this paper in order that the Engineer may satisfy himself of their correctness; the results alone are given in the Text so as not to interfere with the discussion on the practical points involved].

Sub-surface Velocity.

28. The extensive experiments on the Mississippi and its affluents have conclusively shown* that "sub-surface velocity" varies according to such a law that it may be represented by the abscissæ of a parabola, whose ordinates represent depths below a certain line which is generally the line of greatest velocity and *below* the surface.

This will be understood from the velocity-diagram (*Fig.* 2, *Plate* A), which clearly represents to the eye the law of variation with the depth; and it may be noted that *with the same data* of surface, bottom, or intermediate velocities, the parabolic theory will always give the *largest* result for the Total Current-Pressure—it is easily seen from the figure (*Fig.* 2, *Plate* A), that the parabolic area encloses, *i.e.*, is larger than, any area formed *with same data* by simply joining extremities of the lines representing given velocities (*i.e.*, as if the velocity were uniformly-varying).

29. The complete determination of the sub-surface velocities, or, which is the same thing, the construction of the representative parabola requires *three* data, *e.g.*, three observed velocities at three different known depths.

The only observed velocity ordinarily recorded (at any rate for Indian rivers) is the "Surface velocity." It seems convenient to take for the remaining data—

(2). The "bottom velocity," which must be taken from Hydraulic Tables, or assigned by the practical Engineer.

(3). The position of the axis of the parabola, which is given in the Mississippi Report (pages 262 and 288), by formula (13) (below) as depending in a certain manner on the wind and on the "hydraulic mean depth."

With the following notation slightly altered from that in the Mississippi Report (page 200)—

d = any depth below surface.	V = velocity at depth d (required).
D = depth of bed.	V_D = bottom velocity, (assumed).
d' = depth of axis of parabola.	V' = velocity at depth d' (usually the greatest).
o = depth of surface.	V_o = surface velocity, (observed).
r = hydraulic mean depth.	

f = force of Wind, estimated as zero for a calm, and 10 for a hurricane, reckoned positive or negative according as it sets up or down-stream.

* Report on Physics and Hydraulics of the Mississippi, by Capt. A. Humphreys and Lieut. H. L. Abbot, 1861, pages 234, 287.

Then the Mississippi Report gives (pages 262, 288) for the position of the axis—

$$d' = (\cdot 317 + \cdot 06\, f) \,.\, r \quad\ldots\ldots\ldots\ldots (13).$$

Thus for a half hurricane down-stream (f = about $-$ 5), it follows that $d' = o$, *i.e.*, the axis of the parabola (which is also the line of greatest velocity) lies *in the surface.*

[This is an important simplification of formula (13) as will appear below].

30. *Accurate formulæ.*—The following formulæ (for details *see* Art. 70) follow immediately from the parabolic theory:—

$$V = V_o - (V_o - V_D) \,.\, \frac{d\,(d - 2d')}{D\,(D - 2d')} \quad\ldots\ldots\ldots\ldots (14).$$

$$V' = V_o + (V_o - V_D) \,.\, \frac{d'^2}{D\,(D - 2d')} \quad\ldots\ldots\ldots\ldots (15).$$

Approximate formulæ.—For reasons given below, it seems sufficiently accurate *in the present Problem* to assume $V_D = o$, $d' = o$, which reduces formulæ (14) and (15) to the much simpler forms—

$$V = V_o \left(1 - \frac{d^2}{D^2}\right) \quad\ldots\ldots\ldots\ldots (16).$$

$$V' = V_o \quad\ldots\ldots\ldots\ldots (17).$$

[The Velocity-diagram (*Fig.* 2, *Plate* A) has been constructed from formula (16)].

31. *Explanation of assumptions* $V_D = o$, $d' = o$.—It is probable that the portion of bed scoured out near the Piers is often limited to a saucer or funnel-shaped hollow round the Pier, inside which there will be sometimes violent eddying and boiling action; out so long as this violent action lasts, *scour is going on.* The most unfavourable time for the Pier appears to be, when the scour has reached its full depth, after which time the eddying and surging action must be comparatively small, as, by hypothesis, the scour is not increasing. It seems further probable that the most unfavourable time to the Pier will be when this scour has been *very extensive in the neighbourhood of the Pier*, so that the Pier is exposed to a Current of the full depth of scour, that is to say to a Current whose bed has been lowered to scour level, and which is *throughout its depth effective as a Current in pressing on the Pier.*

This hypothesis enables the whole effect of the current to be reduced to a Statical Problem of Pressure; the present state of Hydraulic Science does not enable the effect of the eddying and plunging action to be in any, beyond a hypothetical, way included, but if this hypothesis be admitted as *the most unfavourable to the Pier*, it appears unnecessary further to consider such eddying action.

Under this hypothesis the depth of river-bed, and of scour in the neighbourhood of the Pier, are considered as the same: this must of course be one* of the data, either from actual observation or assigned by the Resident Engineers.

32. But, further, the Bottom velocity (V_D) seems (to the author) likely to be so

* It is a remarkable instance of the imperfection of the practical data for this Problem that this is one of the elements involved in doubt.

small *for a quicksand bed*, (the case in hand),—perhaps that which will just disturb loose sand—compared with the Surface velocity (V_o), that it might be neglected in forming an *approximate* formula.

[*N.B.*—The ordinary Hydraulic Tables *profess* to give the "Bottom velocity" corresponding to various "surface velocities," but the fact is that such Tables are constructed from formulæ in which the "constants" were derived chiefly from experiments on artificial conduits. The results appear (to the author) totally inapplicable to quicksand *Ex.* The "Surface velocity" in the Rámgangá has been recorded as 16 feet (= 192 inches) per second. By Du Buat's formula the corresponding "bottom velocity" would be $(v - 2\sqrt{v} + 1)$ inches, *i.e.*, 165·3 inches per second—*no matter what the material of the bed was.* Is this possible in a quicksand bed? The bottom would be surely scoured away].

33. *Further simplification $d' = o$.*—It will be observed (Eq. 14) that the sub-surface velocities (V) depend both on the surface velocity V_o and on d', and therefore (by Eq. 13) on f, the force of the wind. Now the objects of the present inquiry are to ascertain both the Greatest Current-Pressure, and the Greatest Moment of Current-Pressure on the Pier.

It is evident that the sub-surface velocities (V) increase with the surface velocity (V_o) which is itself re-inforced by a strong down-stream wind. On the other hand, a down-stream wind (f negative), raises the axis of the parabola (*see* Eq. 13), and thereby diminishes the values of the sub-surface velocities (V) for a given surface velocity (V_o).

The Problem of finding the really Greatest Current-Pressures and Greatest Moments of the same under these conflicting circumstances seems strictly insoluble, as the relation between V_o and f is unknown, but it seems almost certain that "a down-stream wind is more effective in increasing the sub-surface velocities by its increasing the surface velocity (V_o) than effective in decreasing them in consequence of its raising the axis of the parabola."

The adoption of the value (f = about − 5) corresponding to a down-stream half-hurricane causes so great a simplification of the formulæ (14) and (15) and of all others thence derived for Total Current-Pressure, and Moment of the same, (in consequence of its making $d' = o$), that there would be great advantage in adopting it if it could be shown to be sufficiently approximate.

Now it will be found, by actually constructing the parabola, that a variation in the value assigned to *a down-stream* wind *greater than a half-hurricane* (even if from a half to a whole hurricane $f = -5$ to -10) causes *very little change* (diminution) in the sub-surface velocities for a given surface velocity.

Thus the assumption $d' = o$ causes only *a slight over-estimation* of the values of V (sub-surface velocities), so that the simple formulæ (16) and (17) may be regarded as *good approximations* for the *present problem.*

34. *Current-pressure.*—Our knowledge of current-pressure is wholly empirical; it is approximately represented* by the formula :—

$$p = k\,w\,a\,\frac{V^2}{2g}, \quad \text{...(18).}$$

$$\left.\begin{array}{l}\text{Total Current-pressure}\\ \text{on an area } a \text{ of } \textit{small}\\ \text{depth,}\end{array}\right\} = k \times \left\{\begin{array}{l}\text{"heaviness" of fluid}\\ \times \text{ area of cross-section,}\\ \times \text{ height due to velocity,}\end{array}\right\} \quad \text{................(18).}$$

* Rankine's Applied Mechanics, Art. 653.

where k is a quantity depending on the shape and material of the body —at present only obtainable from experiment, and on which unfortunately few experiments are available.

[In works on elementary Hydro-dynamics, (*e.g.*, Cape's Course of Mathematics, Vol. II., Art. 546), the value of k for a cylinder is stated to be $k = \frac{3}{5}$ or ·6, but this includes only the effect of "impact," and is based on a "theory" of current fluid, which is admitted to be most imperfect.

The best value obtainable for the present problem appears (to the author) to be $k =$ about ·8, and for the following reason:—Professor Rankine states* that "the co-efficient k is less for a solid moving in a fluid, than for a fluid moving past the same solid," also that "for a cylinder moving sideways, $k =$ about ·77"].

In absence of better data, the author suggests that for the present problem (a current flowing past a fixed cylinder) the value $k = {\cdot}8$ should be adopted as being a simple figure and *higher* than ·77 as required.

[The Pressure-diagram for current-pressure, *Fig.* 3, *Plate* A., has been constructed from Formula (18). It clearly shows the variation of Current-pressure with the depth; the diminution of Current-pressure on the Pier due to its being 3 feet narrower than the Well is very evident].

35. Then it may be shown by an easy integration (*see* Art. 71), that if

V_o = Surface Velocity in feet per second.
P = Total Current-pressure parallel to stream in pounds.
D = Depth of current in feet.
A = Area of vertical diametral section immersed in square feet.

Then for a Well *tapering only slightly*,

$$P = \frac{8}{15} \cdot kw\,A \cdot \frac{V_o^2}{2g} \text{ (approximately)}, \dots\dots\dots\dots\dots\dots (19).$$

Also in cases where a Pier is placed on a Well of considerably larger diameter, and both are either cylindric or taper very slightly, if

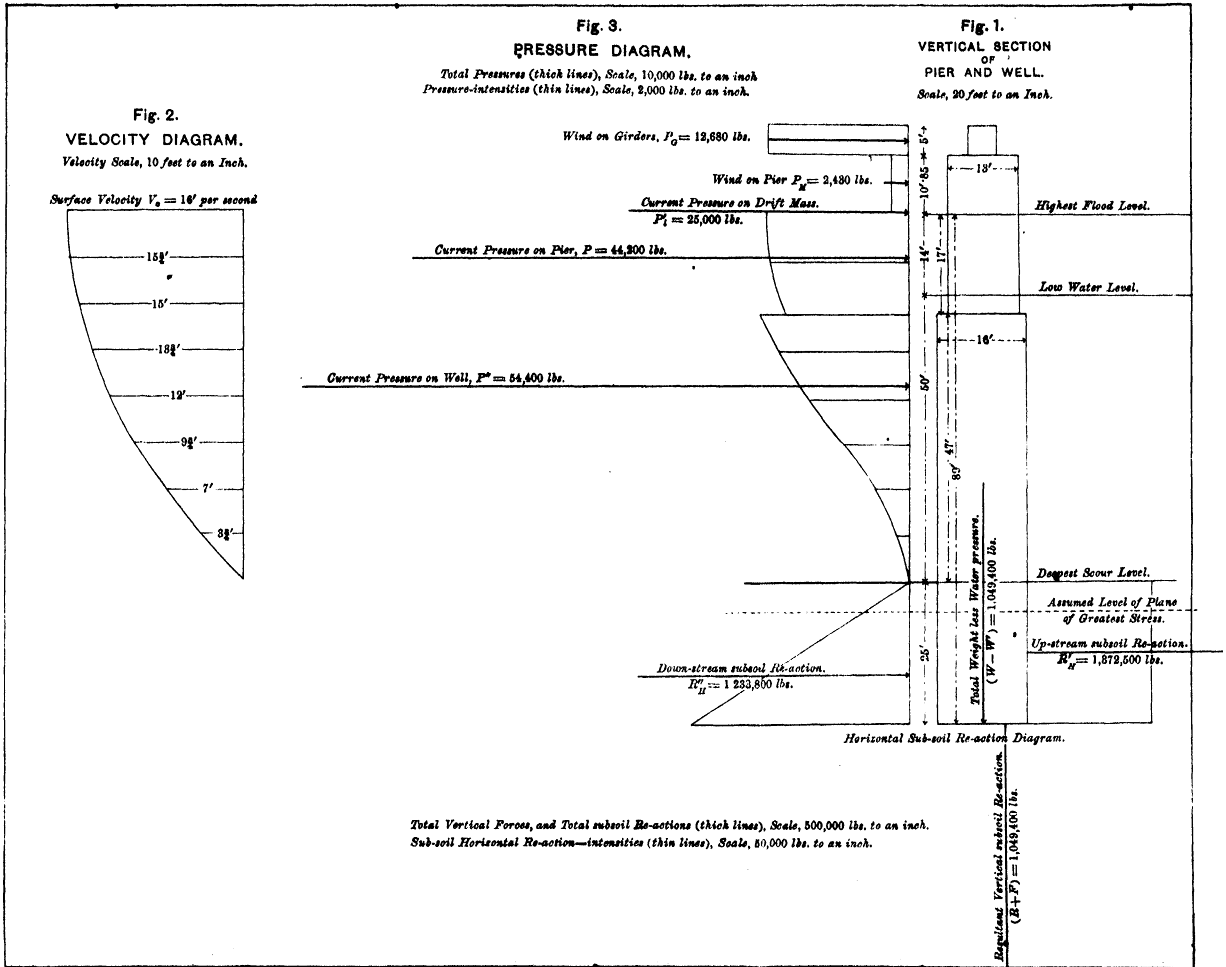
P' = Total Current-pressure parallel to stream, on the Pier.
P'' = Total Current-pressure parallel to stream, on the Well.
d = Depth of Pier immersed.
A' = Area of vertical diametral section of Pier immersed.
B = Mean breadth of Well.

$$\lambda = \left(1 - \tfrac{2}{3}\,\frac{d^2}{D^2} + \tfrac{1}{5}\,\frac{d^4}{D^4}\right) \dots\dots\dots\dots\dots\dots (20).$$

$$\left.\begin{aligned} \text{Then } P' &= kw\,A' \cdot \frac{V_o^2}{2g} \cdot \lambda, \text{ (approximately)}, \\ P'' &= kw \cdot \frac{V_o^2}{2g} \cdot B\left\{\frac{8}{15}\,D - \lambda d\right\}, \text{(approximately)}, \\ P &= P' + P'' \end{aligned}\right\} \dots\dots (21).$$

N.B.—If d be small compared with D, the quantity λ may be taken

* Rankine's Applied Mechanics, Art. 653.



Litho., T. C. Press. Roorkee.

as $\lambda = 1$ approximately : this is an important simplification for numerical calculation.

[The Total Current-pressures on the Pier and Well, *Fig.* 1, are of course represented by the areas of the representative Diagrams, *Fig.* 3, *Plate* A, of Current-pressure].

36. *Centre of Current-pressure.*—It is convenient to find this point as, once found, the estimation of the Moment of Current-pressure about any chosen point whatever is an easy problem of elementary Statics. It is shown in the Appendix to this paper that if

d_o = Depth of centre Current-pressure on whole Well.
d_o' = " " " Pier only.
d_o'' = " " " Well only.

Then in case of a Well *tapering only slightly*

$$d_o = \frac{5}{16} D, \text{(approximately)}, \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

And in case of a Pier on a Well of considerably larger diameter, neither tapering except very slightly,

$$\left.\begin{aligned} d_o' &= \frac{\mu}{\lambda} \cdot \frac{d}{2}, \text{(approximately)}, \\ d_o'' &= \frac{1}{2} \cdot \frac{\frac{1}{3} D^2 - \mu d^2}{\frac{8}{15} D - \lambda d} \text{(approximately)}, \end{aligned}\right\} \ldots\ldots\ldots\ldots (23).$$

$$\text{Where } \mu = \left(1 - \frac{d^2}{D^2} + \frac{1}{3}\frac{d^4}{D^4}\right), \lambda = \left(1 - \frac{2}{3}\frac{d^2}{D^2} + \frac{1}{5}\frac{d^4}{D^4}\right) \ldots\ldots (24).$$

Also if d be small compared with D, then $\lambda = 1$, $\mu = 1$ (approximately) so that these formulæ become

$$\left.\begin{aligned} d_o' &= \frac{d}{2} \text{(approximately)}, \\ d_o'' &= \frac{5}{16}\left(D + \frac{15}{8} d\right) \text{(approximately)}, \end{aligned}\right\} \ldots\ldots\ldots\ldots (25).$$

37. *Moment of Current-pressure.*—By elementary Statics, if

M = Total Moment of Current-pressure in foot-pounds.
M′ = Moment of Current-pressure on Pier only in foot-pounds.
M″ = Moment of Current-pressure on Well only in foot-pounds.
H = Assumed depth (below surface of current) of *any* point chosen as "centre of moments."

Then in case of a Well tapering *only slightly*,

$$M = P.(H - d_o) = kwA \cdot \frac{V_o^2}{2g} \cdot \left(\frac{8}{15} H - \frac{1}{6} D\right) \ldots\ldots (26).$$

And in case of a Pier on a Well of considerably larger diameter neither tapering, *except very slightly*

f

$$\left.\begin{aligned} M' &= P'.(H - d_o') \\ M'' &= P''.(H - d_o'') \\ M &= M' + M'' \ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(27)}.$$

Impact of Drift.

38. It has been explained (Art. 9) that it is impossible to calculate the effect of Impact *properly:* an imperfect equivalent—believed however to be sufficient for the purposes of Engineering—was proposed in the substitution of a Mass of floating Drift supposed as caught by the Pier, and exposed to the full force of the current.

Let A_1 = assumed area in square feet of vertical cross section of supposed floating Drift exposed to full force of current.

V_o = surface velocity.

P_1 = Total Current-pressure on area A_1.

Then as the depth of the supposed Drift-Mass is necessarily small, compared with the depth of the river, and as the shape and size of that Mass are entirely arbitrary, it is permissible—

1°. To assign any arbitrary value to the quantity k in the formula of Art. 34; it is convenient to assume $k = 1$.

2°. To disregard the variation of velocity in the small depth of the supposed Drift-Mass, so that the mean velocity of impinging current may be assumed as sensibly equal to the surface velocity (V_o).

Thus under these premisses, there results for the Total Current-pressure on the Drift-Mass,

$$P_1 = w \,.\, A_1 \,.\, \frac{V_o^2}{2g}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(28)}.$$

39. *Moment of Impact.*—Under the same premisses the following will be a *sufficiently approximate* value for the Moment of that Current-pressure about a point H feet below the current surface, *viz.*,

$$M_1 = P_1 \times H = w \,.\, A_1 \,.\, \frac{V_o^2}{2g} \,.\, H, \ldots\ldots\ldots\ldots\ldots \text{(29)}.$$

40. *Horizontal Re-actions.*—The "Disturbing Forces" develop Horizontal Re-actions in the subsoil of two kinds—

1°. Direct, being the direct normal pressure round the Pier.

2°. Tangential, *i.e.*, horizontal friction between the subsoil and base of the Pier.

Let R_H', R_H'' be the sum of the resolved parts of the Direct Normal Pressures, resolved parallel to the stream, against the down-stream and up-stream (semi-cylindric) sides of the Pier respectively, so that

R_H' = Total up-stream "Direct" Re-action.

R_H'' = Total down-stream "Direct" Re-action.

F_H = Total horizontal friction.

Then at all times, when there is no motion, this equation must obtain

$$\left.\begin{array}{c}\text{“ Sum of horizontal Re-actions} = \text{Sum of Disturbing Forces,”}\\ \text{or } R_n' - R_n'' + F_n = P_e + P_w + P + P_1 = \Sigma(P),\end{array}\right\} (30).$$

It may be assumed that F_n is very small compared with R_n', R_n'' : so that neglecting F_n,

$$R_n' - R_n'' = \Sigma(P), \text{ .... } (31).$$

N. B.—The hydrostatic pressure of the Water against the down-stream side of the Pier is omitted from the above enumeration of Resistances, because from the manner in which the experiments on Current-pressure (by which the value of k in formula 18 was determined) were conducted, the resulting formula for Current-pressure expresses only the *excess* of the Current-pressure above the Hydrostatic Pressure against the down-stream side of the Pier.

41. In order to investigate the distribution of, magnitude of, and position of Centre of Pressure of these Re-actions, consider separately the cases

1°. Of a Pier not under the action of Lateral Disturbing Forces.

2°. Of a Pier under the action of Lateral Disturbing Forces.

1°. *Pier not under Lateral Disturbing Forces.*—In this case it is evident that $R_n' - R_n'' = o$, for the Disturbing Forces P are zero and $R_n' = R_n''$, *i.e.*, the Re-actions of the soil are equal up- and down-stream.

[Observing that there is a hydrostatic pressure of D feet of water, being the depth of the current, on the surface of the soil, then if σ = specific gravity of the soil, the "head of water" is equivalent to a "head" of $D \div \sigma$ feet of soil, and according to the usually received theory of loose earth (or quicksand) pressure, the intensity of its pressure against the Masonry is simply proportional to the depth below a plane $D \div \sigma$ feet above the actual surface of the soil, or in fact follows the law of distribution of fluid pressure, so that its "Centre of Pressure" would be the same as that of ordinary fluid pressure against a submerged vertical cylinder whose top and bottom are $D \div \sigma$ and $(D \div \sigma + h)$ feet below the imaginary fluid surface.

It is shown (in the Appendix to this paper) that the height of the Centre of pressure above the base of the Pier is, if $D' = D \div \sigma$,

$$h_o = \frac{h}{3} \cdot \frac{3D' + h}{2D' + h}, \text{ or } \frac{h}{3}\left(1 + \frac{D'}{2D' + h}\right) \text{ .... } (32).$$

It is worth noting that from Eq. (32), h_o is always $> \frac{h}{3} < \frac{h}{2}$; or $h_o = \frac{h}{3}$ when $D' = o$, *i. e.*, when there is "no head", also h_o increases with the "head" (D') approaching to the limit $\frac{h}{2}$ when the "head" (D') is very large].

2°. *Pier under Lateral Forces.*—There are no experimental data for

determining the distribution of, actual magnitude of, and centres of pressure of, the horizontal Re-actions (R_H', R_H'') up- and down-stream, which determination is requisite to the proper solution of the problem: still some useful generalizations may be drawn.

Firstly, so long as there is no actual motion of the Pier, it is obvious that the Equation (31) must hold true,

$$\therefore R_H' > \text{Sum of Disturbing Forces, } \Sigma (P), \ldots\ldots\ldots\ldots (33).$$

It seems most probable that the effect of the Disturbing Forces on the magnitude and distribution of the horizontal Re-actions is as follows:—

1°. To diminish the pressure of the soil on the near (or up-stream) side, and to press the Pier more against the soil on the far (down-stream) side, *i.e.*, to diminish R_H'' and increase R_H'.

2°. To alter the distribution of pressure of R_H', R_H'' in opposite directions, *viz.*,

(*a*). On the near (up-stream) side—diminishing the intensity of pressure near the surface, and increasing it towards the base, and therefore of course *depressing* the "centre" of pressure of R_H''.

(*b*). On the far (down-stream) side—diminishing the intensity of pressure near the base, and increasing it towards the surface, and therefore of course *raising* the "centre" of pressure of R_H'.

These results may be briefly expressed—

"The Disturbing Forces raise and depress the Resultants of the up-stream and down-stream horizontal Re-actions," .... } (34).

These results are very important, as the raising of the Resultant R_H' increases its Moment (of Stability), and depressing R_H'' decreases its Moment (of Instability). Both actions are therefore conducive to Stability.

[It will be seen hereafter, Art. 49, that Stability of Rotation depends chiefly on the power of the soil to bear this alteration of the original distribution of pressure.

There is good reason to suppose that though the first action of the Disturbing Forces may be to decrease R_H'', their ultimate effect must be to increase R_H''. It will be shown that the principal element of Stability (of rotation) is the Moment of Stability of the up-stream horizontal Re-action (R_H'), which may be increased—

1°. By raising its Resultant, so as to increase its leverage.

2°. By increasing its magnitude (R_H').

But the nature of the soil will not admit of increase of the leverage by raising of the Resultant (which involves increased intensity of pressure towards the Surface) beyond a certain limit, so that further increase of Moment of Stability of R_H' must be due to increase in the actual magnitude of R_H'.

But since $R_H' - R_H'' =$ Sum of Disturbing Forces, it is obvious that for a given Disturbing Force an increase in R_H' is accompanied by an equal increase in R_H''].

42. The above general results are probably all that can be certainly affirmed as to the distribution of, magnitude of, and position of the Re-

sultant of the Horizontal Re-actions R_H', R_H'' in absence of experimental evidence as to the nature of the subsoil resistance.

[It will be shown in Art. 53 how on certain hypotheses as to the nature of that resistance, definite numerical results can be obtained].

Lateral Stability—Stability of Sliding.

43. It is clearly essential to the use of the Pier that it should not slide away, as a whole, down-stream. It appears certain that in some of the great rivers with sandy beds, there is motion—of the *water* at any rate—going on for a considerable depth below the visible "bed" of the stream.

[This is an inference from the observed fact that, though the very large quantity of water withdrawn for canals from the large rivers, such as the Ganges and Jumna, greatly reduces the quantity of water in the rivers for some miles below the canal heads, it makes no perceptible difference in those rivers at a distance of many miles from the canal heads].

Whether the subsoil-particles themselves also partake of this motion or not, is not certainly known; it is believed by many that they do.

The author considers that for the purposes of the present problem, the "surface of absolute rest among the subsoil-particles should be considered the *virtual* bed of the stream."

It appears (to the author) essential that Piers should be sunk well below this "virtual bed"—*i.e.*, that the failure of such Piers as cannot be sunk well below the surface of no motion among the subsoil-particles is certain, and will take place by sliding, as a whole, down-stream.

It will be assumed therefore that all Piers will in practice be sunk well below this level.

The Total Force which tends to cause sliding is simply the sum of the "Disturbing Forces": the Total Resistance to sliding is simply the algebraic sum of the Horizontal Re-actions (R_H', R_H'') so that at any moment when motion is not actually going on Eq. (30) must obtain,

"Total Disturbing Force = Total Resistance to sliding."

But there is a natural limit to the latter quantity (Resistance), *viz.*, the "greatest intensity of pressure which the soil can bear with safety at any point." There is no experimental evidence on this point. Without this practical datum it is simply impossible to estimate numerically the "Working Resistance to sliding," meaning thereby the "Greatest Resistance to sliding which the soil can supply with safety."

SECTION IV.—ROTARY STABILITY: STABILITY OF ROTATION.

44. Well-foundations as actually constructed appear to have failed in general, by "tilting over," *i.e.*, by want of "Stability of rotation," so that this Problem assumes particular practical importance.

In order that the Pier may possess "Rotary Stability" as a whole, it is necessary

(1) That the external forces of all sorts, including *all* Re-actions, should be such that in the *normal* position

"The algebraic sum of their Moments about *any* point = o," or, ...(85),

"The Sum of Moments of Stability = Sum of Moments of Instability,".......(85).

(2). Also that the Re-actions should be of such character that on the Pier undergoing a *very small* hypothetic, not actual, displacement (tilting)—

"The Resultant Moment of *all* the external Forces, including *all* Re-actions, should "tend to restore the Pier to its normal position," *i e.*, "Sum of Moments of Stability $>$ sum of Moments of Instability,"..(86).

These conditions may be called—

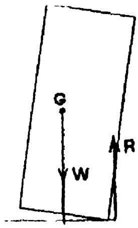
(1). The condition of temporary equilibrium.

(2). The condition of permanent rotary Stability.

45. There is one peculiar difficulty in this Problem (of Rotary Stability) in the present case of quicksand foundations, *viz.*, that the Re-actions of the soil alluded to are of very different character to the ordinary Re-actions of solid or sensibly rigid supports (*e g.*, rock foundations): the latter alone are met with in the usual Problems in Structures, and are alone familiar.

[There is little difficulty in estimating the Moment of Stability of a solid Structure placed on a rigid support; when no lateral (disturbing) Forces are applied, such a Structure presses *all over its base*, so that the Re-action also is distributed over the base, but the instant the Structure is *tilted ever so slightly* by lateral forces, the distributed Re-actions are immediately concentrated into a *single Re-action* passing through the heel, or point of rotation, so that by taking moments about any chosen point on this line, the moment of the Re-action is zero and does not therefore appear in the result, so that no attention need be paid to it. It is in fact commonly simply ignored in ordinary Text-books: the above is, however, the sole justification of thus disregarding the Re-action. It is otherwise with the Re-actions in the present problem: they do not become so concentrated into a single Resultant upon a *slight* tilting of the Structure, so that there is no point about which the Moment of these Re-actions obviously vanishes, and they cannot be disregarded in estimating the Moment of Stability].

Fig. 4.



In fact the question of "STABILITY of rotation" in the present case resembles that of the Stability of Ships to the extent that the Re-actions

are distributed pressures which do not greatly vary for a slight displacement (titling over).

46. It ought to be sufficiently obvious that the Equation of Moments (Eq. 35) must be true for any point whatever, so that it is a matter of indifference, theoretically, what point be chosen as "Centre" for estimating Moments: it may make great difference in *convenience* of calculation what point is chosen, but the *result* will be the same—

e.g., in Structures on solid (sensibly rigid) supports, it is convenient to choose some point on the Resultant Re-action, after the Structure has undergone a slight displacement, simply because the Moment of the Re-action does not appear in consequence in the result—a matter of some convenience for saving calculation—but the choice of any other point would eventually lead to the same result.

But about whatever point the Moments be estimated it is essential to include in the Equation of Moments *all* Re-actions whose Resultants do not pass through the chosen point.

[No difficulty is felt about this in the ordinary problem of Structures on rigid supports because, as explained, the "centre of Moments" can be so chosen as to render consideration of the Re-actions unnecessary. There is in the present problem also no doubt *some* point through which the Resultants of some of the Re-actions pass, but as the position of this point is not known *à priori* it cannot be chosen].

It is, on the whole, perhaps most convenient to choose the extreme lower (down-stream) point of the base as "Centre of Moments," because all "arms of levers" are thus positive.

[A simple consideration will now show the necessity of including *all* Re-actions in the Equations of Moments.

With increased depth of sinking, the arm of lever of the Disturbing Forces increases, and therefore also their Moment (about chosen point), which is of course the Moment of Instability, but the arm of leverage of the Weight of the Pier remains constant, being simply the radius of the Pier's base, so that its Moment, which is a Moment of Stability, remains constant. If the Re actions be disregarded, the following absurd result necessarily follows :—

"Increased depth of sinking is attended with decrease of rotary Stability"]

47. *Moment of Instability.*—This is the sum of the Moments of the Disturbing Forces and of some of the Re-actions.

(1). Wind on Girders.	(5). Upward Water-pressure.
(2). Wind on Piers.	(6). Direct Vertical Re-action of soil.
(3). Current on Drift-Mass.	(7). Vertical Friction.
(4). Current on Piers.	(8). Lateral Resistance of soil on up-stream side.

Moment of Stability.—This is the sum of the Moments of the Weights and of one Re-action.

(9). Weight of superstructure.	(11). Lateral Re-action of soil against down-stream side.
(10). Weight of pier.	

Formulæ have already been given for the Moments of (1), (2), (3), (4); those of (5), (9), (10) are also readily calculable; but the experimental data are entirely wanting for properly calculating the Moments of (6), (7), (8), (11), so that it is impossible to present any certain numerical estimate of the Rotary Stability or Instability of the Piers, as it has been explained to be essential to include (6), (7), (8), (11) in the Equation of Moments. Some useful generalizations may, however, be made.

48. Separating the Moments of vertical from those of horizontal forces, and observing that by Art. 12,

$$\left.\begin{array}{l}\text{Sum of vertical forces pro-}\\ \text{ducing Instability, } viz.,\\ \text{Nos. (5), (6), (7),}\end{array}\right\} = \left\{\begin{array}{l}\text{Sum of vertical forces pro-}\\ \text{ducing Stability, } viz.,\\ \text{Nos. (9) and (10),}\end{array}\right\} \ldots\ldots\ldots\ldots (37),$$

but that the arms of leverage of the Resultants of the former are less than those of the latter, being in the latter case simply the radius of the base, and in the former some *smaller* quantity, it follows that—

"The Resultant Moment of *all* the Vertical Forces, including Weight and Re-actions, is always a Moment of Stability."

Increased depth of sinking does not affect the arms of leverage of Nos. (5), (9), (10), and it seems (to the author) extremely probable that it does not affect the arm of leverage of the remaining vertical forces Nos. (6) and (7). Hence (Eq. 37), it may be inferred with a high degree of probability that—

"The Resultant Moment of *all* the Vertical Forces is always a Moment of Stability, and not affected in amount by increased depth of sinking,"... } ....(38).

49. Next considering the horizontal forces, it is clear that

"Increased depth of sinking increases the Moments of both kinds (Stability and Instability) of the horizontal forces."

Observing that the sum of the Disturbing Forces, (1), (2), (3), (4) is unaffected by increased depth of sinking, also that the Resultant up-stream Lateral Re-action of the soil, *i.e.*, $R_H' - R_H''$, the excess of (11) over (8), is also unaffected by increased depth of sinking, (being equal to the sum of the Disturbing Forces), it follows that the Moment of Instability of the Disturbing Forces is *increased* by increased depth of sinking simply by increased arm of leverage, but on the other hand that the Resultant Moment of Stability developable (though not necessarily developed) from the Lateral Re-action of the soil *is increased in a still higher ratio* than the former, in consequence both of increase in the Total Re-actions deve-

lopable from the increased extent of soil (and that of greater compactness) in contact with masonry, and also of the favourable change, *see* Result (34), in the arms of leverage of the Re-actions. These Results may be briefly expressed,

"Increased sinking increases the Moments of Instability of the Disturbing Forces, and also the available Resultant Moment of Stability of the Horizontal Re-actions—the latter in a higher ratio," .... (39).

[Both these Re-actions, both up-stream and down-stream, be it observed, are *simultaneously* necessary to Stability of Rotation].

This last is a *most important* Result, for in consequence of all the Vertical Re-actions having been proved to be conducive to *Instability*, *vide* Results (3) and (6), it is probable that the "Resultant Moment" of *all* the Vertical Forces, which has been shown to be a Moment of Stability, is *not of great magnitude*, and as it has been explained that it most probably cannot be increased by increased depth of sinking, it follows that—

"The Weight of the Pier is in quicksand comparatively unimportant in producing STABILITY (of rotation), also that .... (40).

"Stability (of rotation) of Piers in quicksand is almost entirely due to the Lateral Resistance of the soil," or in other words ....

"Stability of Rotation can *only* be secured by sinking to a depth at which the required Lateral Resistance can be developed," . ... (41).

50. The utmost that it seems possible to do *with certainty* in the way of *numerical estimation* of Stability of Rotation is to calculate the Moments of Instability Nos. 1 to 5, and of Stability Nos. 9 and 10. Their difference will be a Moment of Instability. It can then only be left to the practical Engineer to judge empirically whether the requisite Moment of Stability No. 11 is or is not developable from the particular subsoil.

51. But by making some hypotheses, guided by knowledge of the soil, as to the nature of the resistance of the soil, more definite numerical results may of course be obtained. Although the results are of course *to that extent* hypothetical, still if the hypotheses are well chosen, the results will be *some* guide to the practical Engineer, and better than no guide at all.

[The usual hypothesis in works* on Applied Mechanics as to the nature of resistance of the soil is that the soil is *capable* of bearing and therefore of yielding pressure of "uniformly-varying" intensity, *i.e.*, varying as the distance from a fixed axis, provided that there be pressure over all parts of the surface pressed, at least equal to the least, and not greater than the greatest, internal pressure (usually called "conjugate pressure") due to frictional stability of the particles].

* *See* Rankine's Applied Mechanics, Arts. 197, 198, 199.

The course proposed is *by assigning some probable manner of distribution* of the various Re-actions to deduce the numerical values of the Total Horizontal Re-actions required from the soil, and of the mean and maximum intensities of the same. It will then rest for the practical Engineer to consider whether the particular soil can or cannot yield the *intensity of pressure* indicated as necessary.

[*N.B.*—The greatest intensity of pressure which the soil can safely bear should of course be one of the practical data to be furnished with a Problem of this sort].

It will be convenient to consider the Moments of the Vertical and Horizontal Forces separately.

52. *Moment of Vertical Forces.*—A probable approximation to the Resultant Moment of Stability of *all* the Vertical Forces, Weights and Re-actions included, may be made by assigning the position of the Resultant of Vertical Re-action of the soil, both Direct and Frictional. It will be remembered that semi-fluidity of the sand is considered to be allowed for in Art. 14. Assuming that the sand at the base is otherwise capable of bearing a uniformly-varying pressure, it seems probable that a limit to *the extent of that variation* is that there should be *some pressure* (no tension) at every point of the base, *i.e.*, that the Pier should *press* everywhere on the soil at its base. The Resultant Pressure on the base must in this case* fall within a distance from the centre of the base of $\frac{1}{4}$ (radius).

Observing, *vide* Art. 21—(5), that the Vertical Friction is at the instant of incipient motion of the Pier confined to the down-stream side of the Pier, and most intense towards the down-stream vertical edge, and assuming that its intensity along any vertical edge of the cylinder is simply proportional to the distance of the edge from the diametral plane which lies across the stream, its distribution would be graphically represented by a very thin semi-circular wedge; the Resultant of Forces so distributed lies† at a distance $\frac{1}{4}\pi r = {}^{\cdot}7854\, r$ or $\frac{3}{4}$ radius approximately from the axis. On the above assumption then, the Resultant Vertical Re-actions of the soil are

$$\left.\begin{array}{lll}\text{Direct (R) at a distance} & \frac{1}{4}\,r & \text{from centre of base,}\\ \text{Frictional (F)}\quad ,, & \frac{3}{4}\,r & \quad ,, \quad ,,\end{array}\right\}\text{.........(42).}$$

It may therefore be assumed with considerable confidence that the Resultant Vertical Re-action of the soil (R + F) lies between the distances

* Rankine's Applied Mechanics, Art. 208.

† *See* Appendix, Art. 74.

$\frac{1}{4}r$ and $\frac{3}{4}r$ from the centre of base, or is roughly speaking about $\frac{1}{2}r$ distant from the same.

Now the magnitude of $(R + F)$ is known by Eq. 2, *viz.*, $R + F = W - W'$, a known quantity.

$\therefore$ Moment of Vertical Re-actions of soil = Re-action $\times$ arm of lever,

$$= (R + F) \cdot \frac{r}{2}, \text{ or } (W - W')\frac{r}{2}, \ldots\ldots\ldots\ldots \text{(43)},$$

as a *rough* approximation.

Hence the "Resultant Moment of Stability of *all* Vertical Forces" is *as a rough approximation*—

M_v = Moment of Weight − Moment of Water-pressure − Moment of Vertical Re-actions of soil,

$$= W.r - W'r - (R + F)\frac{r}{2}$$

$$= (W - W')r - (W - W')\frac{r}{2}$$

$$= (W - W')\frac{r}{2} \ldots\ldots\ldots\ldots \text{(44)}.$$

It is worth noting that had the Pier been resting on a *rigid* support (*e g*, rock-foundation), the Moment of Stability of Vertical Forces would have been simply Wr, *i.e.*, the Moment of the Weight of the Pier, and that in this problem, the Water pressure on the base reduces the "effective Weight" of the Pier to $(W - W')$, *see* Art. 16, and that the peculiar *distribution* of the Re-actions reduces the leverage from r to $\frac{1}{2}r$, so that the Resultant Moment is only $\frac{1}{2}(W - W')r$. This bears out the remark that the Pier's mere Weight must not be relied on for producing Stability of rotation, *see* Result (40).

53. *Moment of Horizontal Forces.*—A rough approximation may now be made to the magnitudes of the Total up- and down-stream Horizontal Re-actions (R_H', R_H'') which the soil must be capable of supplying against the Pier, by making some hypothesis as to the *distribution* of those Resistances.

It has been explained (Art. 41, Result 34) that the effect of the Disturbing Forces on these Re-actions was to alter the distribution of both of them, raising the Resultant (R_H') of up-stream Re-action and depressing the Resultant (R_H'') of down-stream Re-action. Now it seems probable that the natural limit to this action will be such an alteration in the distribution of the pressure that—

1°. On the up-stream side, the soil shall *just cease to press* against the Masonry at the surface of the soil.

2°. On the down-stream side, the soil shall press *uniformly*.

Assuming that the vertical distribution of pressure is still "uniformly-varying" in the former case, the effect would be—

1°. To depress the Resultant (R_H'') of down-stream Horizontal Re-action to a height of $\frac{1}{3}\times$ depth (of sinking) above the base, by the usual rules of uniformly-varying pressure.

2°. To raise the Resultant (R_H') of up-stream Horizontal Re-action to a height of $\frac{1}{2}\times$ depth (of sinking) above the base.

[The exact variations of pressure proposed are of course hypothetical, but they are of the character indicated as *certain* in the generalizations in Art. (41)].

Let h = depth of sinking.

$\Sigma(P)$ = Sum of Disturbing Forces.

M_V = Resultant Moment (of Stability) of *all* Vertical Forces.

M_H = Sum of Moments (of Instability) of Disturbing Forces.

M_H' = Moment (of Stability) of up-stream Horizontal Re-action (R_H').

M_H'' = Moment (of Instability) of down-stream Horizontal Re-action (R_H'').

Then by the Results 1° and 2° of above assumptions—

$$M_H' = R_H' \times \frac{h}{2} \quad\text{.. (45).}$$

$$M_H'' = R_H'' \times \frac{h}{3} \quad\text{.. (46).}$$

Hence from the Equation of Moments

$$R_H' \times \frac{h}{2} - R_H''.\ \frac{h}{3} = M_H - M_V \quad\text{.................. (47).}$$

Also from the condition of equality among the horizontal forces

$$R_H' - R_H'' = \Sigma(P) \quad\text{.......................... (48).}$$

54. *Total Horizontal Re-actions* (R_H', R_H'').—As M_H, M_V, P are known quantities, the two quantities R_H', R_H' the up- and down-stream Horizontal Re-actions can be calculated from these two equations. Thus,

$$\left.\begin{aligned} R_H' &= \frac{6(M_H - M_V)}{h} - 2\,\Sigma(P) \\ R_H'' &= \frac{6(M_H - M_V)}{h} - 3\,\Sigma(P) \end{aligned}\right\} \quad\text{.................. (49).}$$

55. *Intensity of Horizontal Re-actions.*—On the foregoing assumptions as to the distribution of pressure, it follows that if

$$\left.\begin{aligned} p_o' &= \text{Mean intensity of up-stream horizontal Re-action,} \\ p_o'' &= \quad \text{,,} \quad \text{,,} \quad \text{down-stream} \quad \text{,,} \quad \text{,,} \\ p'' &= \text{Maximum intensity of down-stream} \quad \text{,,} \\ h &= \text{depth of sinking. ....} \end{aligned}\right\}\text{in pounds per square foot.}$$

Then (by the pressure-distribution assumed in Art. 53, *see* also Art. 56).

$$\frac{R_H'}{h} = \text{Intensity of } R_H' \text{ per vertical foot, ... (50).}$$

$$\frac{R_H''}{h} = \text{Mean Intensity of } R_H'' \text{ per vertical foot, ... (51).}$$

$$2\,\frac{R_H''}{h} = \text{Maximum Intensity of } R_H'' \text{ per vertical foot, ... (52).}$$

$$\left.\begin{aligned}\therefore p_o' &= \frac{R_n'}{hr}\ \ldots\\ p_o'' &= \frac{R_n''}{hr}\ \ldots\\ p'' &= \frac{2R_n''}{hr}\ \ldots\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53).$$

These quantities, (p_o', p''), *the actual greatest intensities of pressure* of the Masonry on the subsoil produced by the Disturbing Forces, under the assumed distribution of pressure, are *the final quantities proposed for calculation* in Art. 51. These having been calculated it rests with the practical Engineer to decide whether any particular soil can or cannot bear them. This of course can only be ascertained by experiment.

56. *Explanation of Results* (50) *to* (53).—Results (50) and (51) seem obvious. Result (52) follows from (51), because the distribution of R_H'' had been assumed as "uniformly-varying," and its intensity at soil-surface zero. Now the maximum intensity is known in such a case to be twice the mean-intensity: hence Result (52).

Results (50) to (52) give the values of the sum of the resolved parts parallel to the stream of the Normal Pressures on a vertical semi-cylinder of one foot in height. The intensity of Normal Pressure causing that Total Pressure (50 to 52) in one direction is known* to be (53).

Example.

57. The following numerical Example is that of a Pier for the Bridge over the Rámgangá, *see* Art. 2.

Fig. 1 shows a vertical section through the Pier's centre of gravity.

Fig 2 is a "Velocity-Diagram," showing the variation of sub-surface-velocity throughout the depth calculated by Eq (16).

Fig 3 is a "Pressure-Diagram" The distribution of the Horizontal Pressures of all kinds is shown by the abscissæ of the figure in thin lines. the Total or Resultant Pressures of all kinds, both vertical and horizontal, are shown by thick lines, and are drawn through the respective Centres of Pressure.

The Vertical Force representative lines necessarily overlap *Fig.* 1.

The use of *several* scales in this Diagram was unavoidable owing to the very great difference in the Forces, thus the scales are as follows :—

For Distribution of Disturbing Forces, ...	2,000 lbs. to an inch.
„ Total or Resultant of Disturbing Forces, ...	10,000 „ „ „
„ Distribution of Horizontal Subsoil Re-actions,	50,000 „ „ „
„ Total or Resultant Horizontal Subsoil Re-actions, ... }	500,000 „ „ „
„ Vertical Forces, ... }	

* Rankine's Applied Mechanics, Art. 179.

TABLE I.—VERTICAL FORCES.

	Heaviness in lbs per c. ft.	Volume in c ft.	Weight in lbs.	Levers e in feet	Moments in foot-pounds.	Reference to Text.
Girders, ...	.	...	44,800			
13 feet Pier to flood-level, ...	112·5	$\pi \times \left(\frac{13}{2}\right)^2 \times 10{\cdot}85$	164,300			
13 feet Pier (in water), ...	50	$\pi \times \left(\frac{13}{2}\right)^2 \times 17$	113,300	...	...	Art. 16.
16 feet Well (in water), ...	50	$\pi \times \left(\frac{16}{2}\right)^2 \times 72$	727 000	..	...	Art. 16.
∴ Resultant of Weight and Water-pressure		i.e., $(W - W') =$	1,049,400	8	8,395,200	Art. 16.
∴ Total Vertical Re-action of soil, i.e., $(R + F) =$			1,049,400	4	4,197,600	Arts. 19, 52, Eq (2),(43).
∴ Result Moment (of Stability) of *all* Vertical Forces, i.e., $M_v =$					4,197,600	Art. 52, Eq. (44).

TABLE II.—DISTURBING FORCES.

		Force.	Pressure-intensity in lbs. per sq foot.	Area in square feet.	Total Pressure in lbs	Leverage in feet	Moments in foot-pounds	Reference to Text.
P_G	Wind	on Girders,	40	317	12,680	$102\frac{1}{2}$	1,297,700	Arts 23, 24, 26, 46. Eq (8), (9), (11).
P_M	Wind	on Pier,	$\frac{1}{2} \times 40$	$121\frac{1}{2}$	2,430	94·4	229,400	Arts 23, 24, 26, 46. Eq. (9), (12).
P_1	Current	on Drift-Mass,	$62·5 \times \frac{16^2}{64·4}$	100	25,000	89	2,225,000	Arts. 38, 39, 46. Eq. (28), (29).
P'	Current	on 13 feet Pier,	$·8 \times 62·5 \times \frac{18^2}{64·4}$	17×13	44,200	$89 - \frac{1}{2} \times 17$	3,566,700	Arts 34, 35, 37, 46. Eq (21), (25), (27).
P''	Current	on 16 ft. Well,	$·8 \times 62·5 \times \frac{16^2}{64·4}$	$\left(\frac{8}{15} \times 64 - 17\right) \times 16$	54,400	$89 - \frac{5}{16}\left(64 + \frac{15}{8} \times 17\right)$	3,753,600	Arts 34, 35, 37, 46. Eq. (21), (25), (27).
		∴ Total Disturbing Force, *i.e.*, $\Sigma(P) =$			138,710			
		∴ Resultant (up-stream) Horizontal Re-action of soil, *i.e.*, $R_H' - R_H'' =$			138,710			Arts. 40, 53. Eq (31), (48).
		∴ Total Moment (of Instability) of Disturbing Forces, *i.e.*, $M_R =$					11,072,400	Art. 46.

Tables I. and II. contain details* of the calculation of all the quantities indicated in this Paper, ending with the following :—

Total Disturbing Force,	$\Sigma(P) =$	138,710 lbs.
Moment (of Instability) of Disturbing Forces,	M_H	= 11,072,400 ft.-lbs.
Moment (of Stability) of all Vertical Forces,	M_V	= 4,197,600 „

Hence are calculated by Eq. (49), Art. 54, the final quantities,

Total Horizontal subsoil Re-actions, { up-stream, $R_H{}' = 1,372,500$ lbs. / down-stream, $R_H{}'' = 1,233,800$ „

also by Eq. (53), Art. 55,

Mean intensity of *up-stream* horizontal subsoil Re-action, ... } $p_o{}' =$ 6,860 lbs. per sq. ft.

Greatest intensity of *down-stream* horizontal subsoil Re-action at foot of Well, ... } $p'' = 12,340$ lbs. per sq. ft.

Approximate calculations were considered sufficiently accurate for this Problem : thus in Formulæ 20, 21, 23, 24, λ, μ have each been taken as $= 1$, also $2g = 64\ 4 =$ 64 nearly. The Results are all stated in round numbers. The numerical work has been carefully checked.

[It rests solely with the Resident Engineers to decide whether the soil can or cannot bear these intensities of pressure safely, and the *Stability* depends solely on this].

Section V.—Transverse Strength.

58. This is a question entirely distinct from that of Stability, and requiring distinct treatment.

The general effect of the Disturbing Forces is to produce two distinct Strains and Stresses at every cross-section, *viz.*, (1), Shearing; (2), Longitudinal (in the present case vertical).

[It is considered that a solid Masonry Structure is necessarily able to bear the Shearing Force, so that the only question requiring treatment is that of the Longitudinal Stress produced].

59. The Weight of the Superstructure and upper courses of masonry produce in a symmetrical Pier (the usual kind) an *approximately uniform pressure* all over each horizontal course of masonry, *when the Pier is not under the action of the Disturbing Forces.*

It may be said in a general manner that the effect of the Disturbing Forces is solely to *alter the distribution* of that pressure, so as to produce a *varying pressure*—least intense on the near (up-stream) side, and most intense on the far or down-stream side.

This variation of pressure may increase with increase of the Disturbing Forces to such an extent as wholly to relieve the pressure on the near side, or even produce tension on that side. The material will fail if either of the two stresses, *viz.*, tension on the near side, or pressure on the far side, exceed the limits of resistance to tension and crushing of the material.

* With list of references to all the formulæ required.

60. The Problem admits of consideration from two aspects—

(1). To ascertain the actual Longitudinal Stress produced at any Cross-section in a given Pier by a given Disturbing Force.

(2). To ascertain the diameter of Pier required to bear with safety the greatest Longitudinal Stress produced by a given Disturbing Force.

It is supposed (by the author) that Piers would in practice be designed from considerations other than those of Transverse Strength, *e g.*, from considerations of Stability, so that the usual problem would be the former (1). Although somewhat more difficult than the latter, it will accordingly be considered.

[It is assumed by general consent* of the profession that the varying Longitudinal Strain produced over a Cross-section by Transverse Strain is a uniformly-varying Strain, also that within elastic limit Stress varies as Strain.

These assumptions are necessary* to the formulæ (54) of Art. 65].

61. *Plane of greatest Stress.*—Remembering that Indian Well-foundations consist in general of a Pier placed on a somewhat larger Well, neither of which taper much, it follows that—

(1). The available power of Resistance to the Longitudinal Stress produced by the Disturbing Forces is approximately the same at every horizontal section of the Well.

(2). The Longitudinal Stress actually produced at any horizontal section of the Well will depend entirely on the actual Bending Moment of the External Horizontal Forces, including Disturbing Forces and Re-actions.

Now the Bending Moment of the Disturbing Forces clearly increases in magnitude from the top of the Well downwards; also under any conceivable distribution of the subsoil Re-actions (Arts. 41, 53) which will produce STABILITY of ROTATION, it seems (to the author) essential that within a short depth of the current bed there should be considerable "horizontal up-stream subsoil Re-action," and that it should rapidly increase with the depth: the Moment of such Re-action will oppose that of the Disturbing Forces. Hence the following important Corollaries :—

1°. "The Bending Moment of *all* the Horizontal Forces, whether Disturbing or Re-actional, increases from the top of the Well downwards, and attains a maximum *at some short distance below* the current-bed, and the "Greatest Longitudinal Stress" occurs at that plane."

2°. Inasmuch as the safety of the Structure depends solely on this "Greatest Stress," it is quite unnecessary, as a practical question, to consider any other.

This is an important saving in calculation.

It would be important to discover the exact position of this plane of greatest Stress, but this cannot be done without assigning the precise distribution of the Horizontal Re-actions. Under the uncertainty attending

* Rankine's Applied Mechanics, Art. 306.

any hypothetical assignment of this distribution, it does not seem (to the author) worth while to introduce so complex an investigation. It seems (to the author) most probable that the " plane of greatest Stress " cannot lie *at a depth greater than* one-fifth of the depth of sinking, having regard to the necessity of STABILITY OF ROTATION.

It will be assumed therefore in the remainder of this investigation that the plane of greatest Stress lies at this depth, *viz.*, one-fifth of depth of sinking.

62. *Resistance of Masonry to direct Strain.*—Those materials whose Resistance to tearing and crushing are equal are in general best fitted to resist Transverse Strain.

The available Tenacity of " Brick Masonry " is clearly the *least* of the following quantities:—

1°. Tenacity of the Brick.
2°. Tenacity of the Mortar or Cement.
3°. Adhesion of the Mortar or Cement to the Brick.

The available resistance of " Brick Masonry " to crushing is clearly the lesser of the following quantities:—

4°. Resistance to crushing of the Brick.
5°. Resistance to crushing of the Mortar or Cement.

The " available Tenacity " of ordinary " Brick Masonry is generally far inferior to its " available resistance to Crushing." In fact English practice in designing Masonry Structures set in Mortar is *not to depend at all* on the Tenacity of the mor'ar.*

An extract from a Memo. by B. Leslie, Esq., bearing on the Strength of Mortar used in the Well-foundations of the Oudh and Rohilkhand Railway is appended.

" My own experience of brickwork in India is that as a rule, it is superior to the average of brickwork in England. This is chiefly owing to the excellent quality of the bricks and the use of surkhi instead of sand for making mortar, by which the strength and tenacity of the mortar may become equal to that of the bricks themselves.

" In considering the tenacity of brickwork in a vertical direction, which is necessary, as the lateral strains resolve themselves into vertical strains at the plane of fracture, we are to a very great extent dependent on the quality of the workmanship, and the supervision maintained during construction. In building up new brickwork upon that which has been built for some time, it is essential to remove two or three courses to arrive at 'green work' to which the mortar will adhere, and even then, as there is no vertical 'bond,' we are entirely dependent upon the adhesion of the mortar for vertical tenacity."

* Rankine's Civil Engineering, Art. 262.

An extract from some "General Observations" on the Well-foundations in the same Railway by E. Byrne, Esq., Resident Engineer at Lucknow, bearing on the same point (*viz.*, Strength of Mortar used in the Well-foundations) is appended.

"The tenacity of Mortar * * * * may as a high figure be taken at 50 lbs. per superficial foot. 300 lbs. attainable by the use of certain cements is not a thing of ordinary practice; and if a floating mass carried down by unexpected floods should bear upon new, and hurriedly constructed brickwork, the shock received may quickly prove that the mortar used in putting it together had not attained a tenacity of even 20 lbs. per superficial foot.

"Rough treatment to brickwork should as much as possible be avoided, for disturbance of any kind is prejudicial to the setting of mortar, and its proper adherence to work which it is intended to keep together.

"The oscillating manner in which wells sometimes go down is attended with risk; for if heeling over be excessive, and when the mortar is still fresh, force be employed to regain position, fracture may ensue."

In absence of good data, and after such conflicting statements there appears (to the author) to be no reason to depart from established English practice for Masonry simply set in Mortar.

Admitting the Mortar to be even of superior quality, considerable allowance must be made for the rough treatment a Well is exposed to in the act of sinking, as being likely to impair the tenacity of its Mortar.

63. But some Indian Well-foundations differ in an important point from ordinary masonry in being bonded together vertically by iron ties connecting horizontal curbs. The introduction of these iron ties must so greatly increase the "available Tenacity" of the Masonry as a whole, that this Tenacity will become a very important element of Transverse Strength.

Practical Remark.—It seems to the author that the best result would be obtained, in future structures, by placing a sufficient number of vertical Iron Ties on the side likely to be in Tension (the up-stream side), especially where the Tension would be most severe, *i.e.*, for some distance above and below the level of greatest scour, to make the "available vertical Tenacity" of the Masonry approximately the same as its available Resistance to Crushing.

64. *Wells classed.*—Indian Well-foundations appear therefore to be divisible into two classes as regards the investigation of their TRANSVERSE STRENGTH, *viz.*,

CASE I.—Tied vertically with iron ties. Tenacity approximately equal to Crushing Strength.

CASE II.—Simply set in Mortar. Tenacity to be disregarded.

65. Formulæ for Greatest and Least Stress-intensities (p', p'') at plane of Greatest Stress.

CASE I. *Well with vertical iron ties.*—Assuming as in Art. 64, that the Ties are applied so as to make the available Tearing and Crushing Strength of the Masonry approximately equal—

Practical Conclusion.—The Crushing Strength of good masonry is known to be about 1,000 lbs. per square inch. It follows that—

It may be shown* that if

W = Total Weight above any Cross-section AB.

A = Area of that section.

M_H = Bending Moment of Horizontal Forces at that section.

α, β = Distances of centre of gravity of that cross-section from the down-stream and up-stream sides of the cross-section.

I_0 = Moment of inertia of the cross-section about an axis across the stream through its centre of gravity.

p', p_0, p'' = Greatest, Mean, and Least intensities of Stress over the cross-section.

$$\left.\begin{aligned} \text{Then, } p' &= \frac{W}{A} + \frac{M_H}{I_0}\cdot\alpha, \text{ (pressure)}\ldots\ldots \\ p_0 &= \frac{W}{A}, \text{ (pressure)}\ldots\ldots \\ p'' &= \frac{W}{A} - \frac{M_H}{I_0}\cdot\beta, \begin{cases}\text{pressure or tension according as} \\ \dfrac{W}{A} >< \dfrac{M_H}{I_0}\cdot\beta\ldots\ldots\end{cases} \end{aligned}\right\}\ldots\ldots(54).$$

CASE II. *Masonry simply set in Mortar.*—As, by hypothesis, the Tenacity of the Mortar is not to be relied on, it follows that there should, for safety, be some *pressure* over every part of the Cross-section, so that the Longitudinal Stress at that section should be all of one kind, *viz.*, pressure.

Now it may be proved* that *in this case* (*i.e.*, when the Stress is all pressure), "the Greatest and Least intensities of Stress are *the same* for all material which is capable of bearing a uniformly-varying Strain, within the Elastic limit of Crushing Strain," and may therefore be found by the *same* formulæ (54) as given for "isotropic" material (*i.e.*, material whose extension and contraction under equal Loads are equal).

* *See* "Professional Papers on Indian Engineering," Second Series, No. LXXV., on "Transverse Strain in Pillars," by the present writer. *See* also para. 198, Chapter XI. of this Volume.

[The limitation to the application of formulæ (54) to this Case is of course that p'' must be positive (*i.e.*, represent *pressure*), so that if, on trial, p'' be *negative*, the formulæ (54) are inapplicable, *i.e.*, do not yield the true values of either Stress, but it must be remembered that the fact of p'' becoming *negative* would indicate *tension* over part of the cross-section, in which case the Pillar *must fail by tearing*, it being supposed incapable of resisting tension, or at any rate be dangerously strained].

66. *Application of formulæ* (54).—These formulæ are true for any cross-section. It is practically sufficient (Art. 61) to find the stress-intensities for the cross-section of greatest stress only, which will be assumed (Art. 61) at $\frac{1}{5}$ of depth of sinking below current-bed. All the quantities W, A, M_H, a, β, I_o must of course be taken for that section.

The quantities W, M_H are of course to be calculated by the principles and formulæ of Sections II., III., IV.

The calculation of W presents no difficulty. M_H is strictly the Resultant Bending Moment of all horizontal forces above the plane of greatest stress, and therefore includes a partial Moment of both up-stream and down-stream Horizontal Subsoil Re-actions (R_H', R_H'').

It appears (to the author) that at any such small depth, as chosen, below current-bed both these partial Moments will be very small compared with the large Moment of Disturbing Forces.

The uncertainty as to their distribution makes it unadvisable to increase the complexity of the calculation by attempting to introduce them, if the error consequent on their omission is small. Now this error is not merely small, as explained, but is an error on the side of safety, because it is easily seen that the partial Moment of R_H' must be greater than that of R_H'', and therefore that the omitted Resultant Moment of R_H', R_H'' is a moment of opposite sign to that of the Disturbing Forces, so that the omission is equivalent to an *over*-estimation of the moment M_H, and is therefore an error on the side of safety. It seems therefore sufficiently accurate for the present Problem to estimate M_H as equal to the Moment of the Disturbing Forces only.

67. *Application to cylindric Wells.*—The cross-section being a circle,

$$\left.\begin{array}{l} \text{Area} = A = \pi r^2 \ldots\ldots\ldots\ldots \\ a = r = \beta \ldots\ldots\ldots\ldots \\ \left.\begin{array}{l}\text{Moment of inertia*} \\ \text{about centre,} \ldots\end{array}\right\} I_o = \frac{\pi}{4} r^4 \ldots\ldots\ldots\ldots \end{array}\right\} (55).$$

Hence formulæ (54) become,

$$\left.\begin{array}{l} p' = \dfrac{W}{\pi r^2} + \dfrac{4M_H}{\pi r^3} \ldots\ldots\ldots\ldots \\ p_o = \dfrac{W}{\pi r^2} \ldots\ldots\ldots\ldots \\ p'' = \dfrac{W}{\pi r^2} - \dfrac{4M_H}{\pi r^3} \ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots (54\text{A}).$$

* Rankine's Applied Mechanics, Art. 205.

[*N.B.*—It is convenient in the first instance to take the lineal *foot* and *avoirdupois pound* as units of length and weight throughout.

The resulting quantities p', p_o, p'' will of course be in *pounds* per *square foot*. These should finally be reduced to their equivalents in *pounds* per *square inch*].

68. *Level of no Tensile Stress.*—It appears from actual examples of cylindric Piers in the Oudh and Rohilkhand Railway (*see* Ex. in Art. 69) that there is always *severe Tension* at plane of Greatest Stress; hence the following important results :—

"Masonry *simply* set in Mortar is *unsafe* for Well-foundations," ....(56).

"Well-foundations should be tied vertically with iron-ties about level of current-bed to enable them to bear this Tension safely," .... (57).

Hence an important question arises, *viz.*, to find the level of no Tensile Stress, as the vertical tie-rods should clearly be carried to this height. This would be found by solution of the Equation (54), $p'' = o$, whence for *any* Well $\frac{M_H}{I_o} . \beta = \frac{W}{A}$, ... or in case of a cylindric well $M_H = \frac{1}{4} Wr$, from Eq. (54A).

[This Equation considered as a function of d (depth of plane required) is so complex that it could only be solved by approximation. The expression for M_H would moreover require a re-casting of the formulæ (19 to 21) for Total Current-pressure and (22 to 27) for Moment of Current-pressure, as the "limits" of the integration would in general be different to those used in those formulæ in the actual forms given in the Text. This can be easily done from the general expressions in the Appendix].

As in the Example chosen for illustration (Art. 69), it will be easy to show that the "level of no Tensile Stress" is actually *but little below* the Flood level, and as the Example seems a fairly typical one, it appears hardly worth while to introduce here the complex formulæ for M_H in general.

The following are important practical conclusions :—

"In slender cylindric Masonry Piers exposing a large Girder-surface to Wind, the "level of no Tensile Stress" is very high up,".................. (58).

"Such Piers if *simply* set in mortar are dangerously liable to failure by want of Transverse Strength, (*i.e.*, by opening of the joint under Tension) under effect of *high wind alone*,".. (60).

"Such Piers should be tied with iron tie-rods nearly throughout their length," (61).

These conclusions will appear startling and even opposed to the result of experience that Piers have almost invariably failed *by tilting over*, *i.e.*, by want of Stability of rotation (Arts. 5 and 44).

It will be well therefore to examine carefully the ground of these conclusions. In the first place the investigation from which the fact of

existence of Tensile Stress is deduced by formulæ (54) depends essentially on the hypotheses that—

(1). "Transverse applied Force causes in Masonry a "uniformly-varying," Longitudinal Strain over any cross-section."

(2). "Stress in Masonry is proportional to Strain within the elastic limit."

These are the hypotheses either explicitly or implicitly made by all writers on Applied Mechanics: the author is not aware that there is any experimental evidence of the truth of these hypotheses in the particular case of Masonry. Herein is the weak point of the investigation. They have however the weight of authority, *viz.*, of general adoption by writers* on Applied Mechanics.

On these "hypotheses" it follows necessarily that a solid cylindric Pier is one of the weakest forms as regards TRANSVERSE STRENGTH: for the utmost deviation of the Centre of Pressure on any cross-section—from its centre of figure—of all the External Forces (both Vertical Load and Disturbing Forces) consistent with the non-production of TENSION is only† ⅛ of the diameter, which in a slender cylindric well is of course a small quantity (*e.g.*, only 2 feet in the Rámgangá 16-foot Wells).

Lastly, with regard to the observed fact, that Wells have hitherto almost invariably failed not by cross-breaking, but by want of STABILITY OF ROTATION, the argument above enforced is only to the effect that—

"There will be *some* Tension throughout the greater part of the height of a slender solid cylindric Well, increasing in intensity downwards from about flood level to a little below the current-bed."

[Mr. Bell's Report gives diagrams which clearly show this important fact in the case of nine different Piers of Railway Bridges on the Oudh and Rohilkhand Railway].

It must be inferred that the quality of Mortar and partial employment of vertical iron ties have hitherto sufficed to prevent actual fracture, but *it does not follow* that the state of TENSION was not *dangerous* to Masonry simply set in mortar. As before stated, English practice‡ is that the TENACITY of Masonry simply set in mortar is not to be depended on.

[That the danger of cross-breaking of slender Masonry Pillars under effect of High Wind only is by no means distant may be inferred from the fact that the Ishápur Factory chimney snapped in one of the cyclones of 1864 and 1867, and the Kidharpur Church spire snapped in both those cyclones].

Example.

69. The same example is chosen for illustration of the principles and formulæ of Section V. as in Art. 57.

* *See* Rankine's Applied Mechanics, Arts. 205, 215. Rankine's Civil Engineering, Art. 263.

† Rankine's Applied Mechanics, Art. 205.

‡ Rankine's Applied Mechanics, Arts. 205, 215.

The plane of Greatest Stress is *assumed* at a depth below current-bed of 5 feet or $\frac{1}{8} \times$ depth of sinking for reasons explained in Art. 61.

Tables III and IV. contain details of the calculations of the Load (W) and Bending Moment (M_H) at plane of Greatest Stress with list of references to all formulæ required; the results being $W = 1,637,580$ lbs., $M_H = 7,747,600$ lbs.

Hence by Eq. (54A).

$p' = 8147 + 19272 = \quad 27,419$ lbs. per square foot, (*pressure*).
$p_0 = 8147 \quad = \quad 8,147$ " " (*pressure*).
$p'' = 8147 - 19272 = -11,125$ " " (*tension*).

These are equivalent to—

Greatest intensity of *pressure* $p' = 195$ lbs. per square inch.
Mean intensity of *pressure* $p_0 = 60$ " "
Greatest intensity of *tension* $p'' = 77$ " "

CASE I. *Well with vertical iron ties.*—This Pier is amply strong enough if so tied on up-stream side (about level of plane of greatest stress) that the effective Tenacity and Crushing Resistance are approximately equal.

CASE II. *Masonry simply set in Mortar.*—When the Disturbing Forces simultaneously reach their maxima, the masonry will be *very dangerously strained*, as it has been shown that a large Tensile Stress will fall on the up-stream side at the plane of greatest stress—(the tenacity of the mortar being disregarded).

[*N.B.*—The *Tensile* Stress will not be exactly 77 lbs. per square inch, as formulæ (54A) are not applicable to Case II. when actual Tension exists. The *proper* formulæ for calculating the Stresses in Case II. when actual Tension exists are so complex, that it does not seem worth the labor of calculation].

It having been shown that there is severe Tension at plane of Greatest Stress, it becomes important to find out the level of "plane of no Tensile Stress", as below this level the Masonry should certainly be tied with iron ties.

Now it is easily seen that at the Flood-level,

$W =$ Weight of Girders + Weight of Pier above flood-level,

$$= 44,800 + 112 \cdot 5 \times \pi \times \left(\frac{13}{2}\right)^2 \times 10 \cdot 85 = 206,880 \text{ lbs.}$$

$M_H =$ Moment of Wind on Girder + Moment of Wind on Piers,
$= 12,680 \times 13\frac{1}{2} + 2,430 \times 5 \cdot 4 = 184,300$ foot-pounds.

$$\text{Also by Eq. (54A), } p'' = \frac{W}{\pi r^2} - \frac{4M_H}{\pi r^3} = \frac{Wr - 4M_H}{\pi r^3}$$

$$\therefore p'' = \frac{206,880 \times 8 - 4 \times 184,300}{\frac{22}{7} \times 8 \times 8 \times 8} = 566 \text{ lbs. per square foot } \textit{pressure}.$$

Hence the Least intensity of Stress at Flood-level is 566 lbs. per *square foot*, or 4 lbs. per square inch, and is *pressure*.

The actual PRESSURE is so small that it is obvious that TENSION must ensue at a short distance lower down; in fact on repeating the process at the level of the top of the Well, it appears that at that level there is an actual TENSION of 5 lbs. per sq. in.

It follows therefore that "from some short distance below Flood level there is some vertical TENSION at each cross-section, increasing downwards to some short distance below the Current-bed." This bears out the general "conclusions" in Art. 68.

TABLE III.—VERTICAL LOAD AT PLANE OF GREATEST STRESS.

	Weight in c. ft.	Volume in cubic feet.	Weight in lbs.	Reference to Text.
Girders, ...	...	...	44,800	
13-foot Pier, ...	112 5	$\times \left(\frac{13}{2}\right)^2 \times 27{\cdot}85$	416,070	Arts. 17. 18.
16-foot Well, ...	112·5	$\times \left(\frac{16}{2}\right)^2 \times (47 + 5)$	1,176,710	Arts. 17, 18.
∴ Vertical Load at plane of greatest Stress, *i.e.*, W = ...			1,637,580	Art. 66.

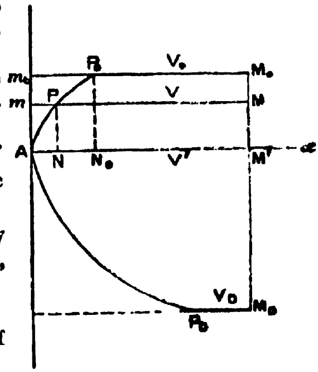
TABLE IV.—BENDING MOMENT AT PLANE OF GREATEST STRESS.

	Pressure-intensity in lbs. per square foot.	Area in square feet.	Total Pressure in lbs.	Leverage in feet.	Moments in foot-pounds.	Reference to Text.
Wind, on Girders, ...	40	317	12,680	$82\frac{1}{2}$	1,046,100	Arts. 23, 24, 26, 61. Eq. (8), (9), (11).
Wind, on Piers, ...	$\frac{1}{2} \times 40$	$121\frac{1}{2}$	2,430	74·4	180,800	Arts. 23, 24, 26, 61. Eq. (9), (12).
Current, on Drift-mass, ..	$62{\cdot}5 \times \frac{16^2}{64{\cdot}4}$	100	25,000	69	1,725,000	Arts. 38, 39, 61. Eq. (28), (29).
Current, on 13-foot Pier, ...	$\cdot 8 \times 62{\cdot}5 \times \frac{16^2}{64{\cdot}4}$	17×13	44,200	$69 - \frac{1}{2} \times 17$	2,674,100	Arts. 34, 35, 37, 61. Eq. (21), (25), (27).
Current, on 16 foot Well, ..	$\cdot 8 \times 62{\cdot}5 \times \frac{16^2}{64{\cdot}4}$	$(\frac{8}{15} \times 64 - 17) \times 16$	54,400	$69 - \frac{5}{16}(64 + \frac{15}{8} \times 17)$	2,121,600	Arts 34, 35, 37, 61. Eq. (21), (25), (27).
Horizontal subsoil Re-actions,	?	?	?	?	omitted as very small	Art. 66.
				$\therefore$ Bending Moment at plane of greatest Stress, *i e*, M_H =	7,747,600	Arts 61, 65, 66.

Appendix.

Construction of formulæ (14), (15) *for sub-surface velocity.*

Fig. 5.



70. The Mississippi experiments show that if $M_0 M_D$ represent the depth (D) of a current, „ $P_0 M_0$ represent the "surface velocity" (V_0), „ $P_D M_D$ represent the "bottom velocity" (V_D), Then the velocity (V) at any point M whose depth below the surface is $M_0 M = d$ will be represented by the abscissa PM of the parabola $P_0 PAP_D$ whose axis is Ax at a distance $d' = M_0 M'$ from the surface depending on the "hydraulic mean depth" and force of the wind.

To find a formula for V the velocity at any depth d in terms of d and of the data V_0, d', V_D, D :—

Let p be the latus rectum of the parabola.

Then $PN^2 = p \times AN$ from the property of the parabola,

or $(d' - d)^2 = p \cdot (V' - V)$ for any point, ...(i).

$\therefore\ d'^2 = p \cdot (V' - V_0)$ for the surface, ... (ii).

and $(d' - D)^2 = p \cdot (V' - V_D)$ for the bed, ... (iii).

Hence subtracting (ii) from (i), $d^2 - 2d'd = p \times (V_0 - V)$.

And subtracting (ii) from (iii), $D^2 - 2d'D = p \times (V_0 - V_D)$.

$$\therefore \frac{d\,(d - 2d')}{D\,(D - 2d')} = \frac{V_0 - V}{V_0 - V_D}$$

Hence $V = V_0 - (V_0 - V_D) \cdot \dfrac{d\,(d - 2d')}{D\,(D - 2d')}$, which is formula (14) of the Text.

Also $V' = V_0 + (V_0 - V_D) \cdot \dfrac{d'^2}{D\,(D - 2d')}$, (for $V = V'$ when $d = d'$), which is formula (15) of the Text.

Total Current Pressure.

71. Case I. Well *cylindrical*, or with *slight continuous taper*.

Let b = breadth of Well at depth d.

b_0, b_D = breadths of Well at surface ($d = o$), and at current bed ($d = D$).

A = area of vertical diametral section of Well in the current.

p = pressure-intensity (per vertical linear unit) round a ring at depth d.

P = Total Current pressure on the Well.

Then $P = \int_0^D p \cdot dd$, (if the cross-section of Well be uniform, or change *continuously*).

$$= \int_0^D kwb \cdot \frac{V^2}{2g} \cdot dd, \text{ by Eq. (18).}$$

$$= \int_0^D kwb \cdot \frac{V_0^2}{2g} \cdot \left(1 - \frac{d^2}{D^2}\right)^2 \cdot dd, \text{ by Eq. (16).}$$

$$= kw \cdot \frac{V_o^2}{2g} \cdot \beta \cdot \int_0^D \left(1 - \frac{2d^2}{D^2} + \frac{d^4}{D^4}\right) \cdot dd,$$ where β is some value of the variable b intermediate to b_o and b_D.

$$= kw \cdot \frac{V_o^2}{2g} \cdot \beta \left[d - \frac{2}{3} \cdot \frac{d^3}{D^2} + \frac{1}{5} \frac{d^5}{D^4}\right]_0^D$$

$$= kw \cdot \frac{V_o^2}{2g} \cdot \beta \cdot D \cdot \left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{8}{15} \times kw \cdot \frac{V_o^2}{2g} \cdot \beta\, D \text{ in general.}$$

Now $\beta = b_o = b_D$ if the Pier be cylindrical.

$\beta = \frac{1}{2}\left(b_o + b_D\right)$ *approximately*, if the Pier taper only slightly.

Hence

$\beta D = A$ in former case, and approximately in latter case.

$\therefore P = \frac{8}{15} \times kw\, A \cdot \frac{V_o^2}{2g}$ for a cylindrical Well, and approximately also for a Well with continuous slight taper. This is Eq. (19) of Text.

CASE II. Short cylindric or *slightly* tapering Pier on a broad cylindric or *slightly* tapering Well.

[If the breadth of Pier at base be very different to that of Well at top, the cross-sections are discontinuous, and the previous formulæ inaccurate. The Total Pressures on Pier and Well must be separately estimated. It is considered that the *slight* taper may in practice be disregarded].

Let P', P'' = Total Current pressures on Pier and Well.

b_o, B = Average breadths of Pier and Well.

d_1 = Depth of top of Well below surface.

Then by a slight modification of steps of Case I.,

$$P' = \int_0^{D_1} p \cdot dd = kw \cdot \frac{V_o^2}{2g}\; b_o \left[d - \frac{2}{3} \cdot \frac{d^3}{D^2} + \frac{1}{5} \frac{d^5}{D^4}\right]_0^{d_1}$$

$$= kw \cdot \frac{V_o^2}{2g} \cdot b_o d_1 \cdot \left(1 - \frac{2}{3} \cdot \frac{d_1^2}{D^2} + \frac{1}{5} \cdot \frac{d_1^4}{D^4}\right);$$ this is Eq. (21) of Text.

$$P'' = \int_{d_1}^D p \cdot dd = kw \cdot \frac{V_o^2}{2g} \cdot B \left[d - \frac{2}{3} \frac{d^3}{D^2} + \frac{1}{5} \frac{d^5}{D^4}\right]_{d_1}^D$$

$$= kw \cdot \frac{V_o^2}{2g} \cdot B \cdot \left\{\frac{8}{15} D - d_1\left(1 - \frac{2}{3} \cdot \frac{d_1^2}{D^2} + \frac{1}{5} \frac{d_1^4}{d^4}\right)\right\},$$ Eq. (21) of Text.

Depth of Centre of Current-pressure.

72. CASE I. Well *cylindric*, or with *slight continuous taper*.

By the ordinary analytical process for finding Centre of Pressure and with notation as in Arts. 35, 36 and 71.

$P \cdot d_o = \int_0^D pd \cdot dd$, (if the cross-section of Well be uniform or change continuously)

$$= \int_0^D kwb \cdot \frac{V^2}{2g} \cdot d \cdot dd \text{ by Eq. (18).}$$

$$= kw \cdot \int_0^D b \cdot \frac{V_o^2}{2g} \cdot \left(1 - \frac{d^2}{D^2}\right)^2 \cdot d \cdot dd \text{ by Eq. (16).}$$

$$= kw \cdot \frac{V_o^2}{2g} \beta \cdot \int_0^D \left(d - 2\frac{d^3}{D^2} + \frac{d^5}{D^4}\right) \cdot dd,$$ where β is some value of b intermediate to b_o, b_D.

$$= kw \cdot \frac{V_o^2}{2g} \cdot \beta \left[\frac{d^2}{2} - \frac{1}{2} \cdot \frac{d^4}{D^2} + \frac{1}{6} \cdot \frac{d^6}{D^4}\right]_o^D$$

$$= \frac{1}{6} kw \cdot \frac{V_o^2}{2g} \cdot \beta D^2.$$

$$= \frac{1}{6} kw \cdot \frac{V_o^2}{2g} \cdot AD,$$ if the Well be cylindrical, also approximately if the Well taper only slightly.

Hence $d_o = \frac{1}{6} kw \cdot \frac{V_o^2}{2g} \cdot AD \div P = \frac{5}{16} D$, which is Eq. (22) of Text.

CASE II.* Short cylindric or *slightly tapering* Pier on a *broad* cylindric or *slightly* tapering Well.

[If the breadth of Pier at base be very different to that of Well at top, the Cross-sections are *discontinuous*, and the previous formulæ inaccurate. The Centres of Pressure on Pier and Well must be found separately. It is considered that the *slight* taper may in practice be disregarded].

By obvious modifications of steps of Case I., and with notation of Arts. 85, 86, 71.

$$P' \cdot d_o' = \int_o^{d_1} pd \cdot dd = kw \cdot \frac{V_o^2}{2g} \cdot b_o \left[\frac{d^2}{2} - \frac{1}{2} \cdot \frac{d^4}{D^2} + \frac{1}{6} \cdot \frac{d^6}{D^4}\right]_o^{d_1}$$

$$= kw \cdot \frac{V_o^2}{2g} \cdot \frac{b_o}{2} \cdot \left(1 - \frac{d_1^2}{D^2} + \frac{d_1^4}{3D^4}\right) \cdot d_1^2$$

$$\therefore d_o' = \frac{1 - \frac{d_1^2}{D^2} + \frac{1}{3} \cdot \frac{d_1^4}{D^4}}{1 - \frac{2}{3} \cdot \frac{d_1^2}{D^2} + \frac{1}{5} \frac{d_1^4}{D^4}} \cdot \frac{d_1}{2},$$ which is Eq. (23) of Text.

$$P'' \cdot d_o'' = \int_{d_1}^{D} pd \cdot dd = kw \cdot \frac{V_o^2}{2g} \cdot B \cdot \left[\frac{d^2}{2} - \frac{1}{2} \frac{d^4}{D^2} + \frac{1}{6} \cdot \frac{d^6}{D^4}\right]_{d_1}^{D}$$

$$= kw \cdot \frac{V_o^2}{2g} \cdot \frac{B}{2} \cdot \left\{\frac{D^2}{3} - d_1^2\left(1 - \frac{d_1^2}{D^2} + \frac{1}{3} \frac{d_1^4}{D^4}\right)\right\}$$

$$\therefore d_o'' = \frac{1}{2} \cdot \frac{\frac{1}{3} D^2 - \left(1 - \frac{d_1^2}{D^2} + \frac{1}{3} \frac{d_1^4}{D^4}\right) \cdot d_1^2}{\frac{8}{15} D - \left(1 - \frac{2}{3} \cdot \frac{d_1^2}{D^2} + \frac{1}{5} \frac{d_1^4}{D^4}\right) d_1},$$ which is Eq. (23) of Text.

Centre of Pressure of Horizontal Re-actions.

78. The depth of the "Centre" of fluid pressure against a vertical rectangle whose top and bottom lines are at depths D' and $(D' + h)$ below the surface is known to be* at a depth below that surface of

$$\text{depth} = \frac{2}{3} \frac{(D' + h)^3 - D'^3}{(D' + h)^2 - D'^2}$$

$\therefore$ Height of that Centre of Pressure above the base is

$$h_o = (D' + h) - \frac{2}{3} \frac{(D' + h)^3 - D'^3}{(D' + h)^2 - D'^2},$$ (which on reduction becomes)

$$= \frac{h}{3} \cdot \frac{3}{2} \frac{D' + h}{D' + h}, \text{ or } \frac{h}{3} \cdot \left(1 + \frac{D'}{2 D' + h}\right);$$ this is Eq. (82) of Text.

* Cape's Mathematics, Vol. II., Art. 424.

Position of Resultant of Vertical Friction.

74. It is explained in Arts. 21 and 52 that at instant of incipient motion the Vertical Friction is confined to the down-stream side of the Pier and that its distribution round that side may be graphically represented by a *very thin* semi-circular wedge. This distance of the Resultant of a system of Forces so distributed from the centre of the wedge is known to be* in general

$$x_0 = \frac{3\pi}{16} \cdot \frac{r^4 - r'^4}{r^3 - r'^3} \text{ where } r, r' \text{ are the external and internal radii of base,}$$

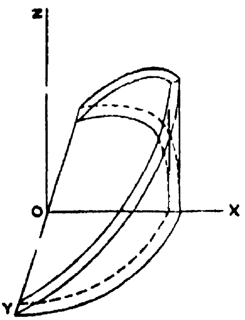
$$= \frac{3\pi}{16} \cdot \frac{r^4 - (r + t)^4}{r^3 - (r + t)^3}, \text{ if } t \text{ be the thickness of shell,}$$

$$= \frac{3\pi}{16} \cdot \frac{4\,r^3\,t + 6\,r^2\,t^2 + 4\,rt^3 + t^4}{3\,r^2\,t + 3r\,t^2 + t^3} = \frac{3\pi}{16} \cdot \frac{4 + 6\,\frac{t}{r} + 4\,\frac{t^2}{r^2} + \frac{t^3}{r^3}}{3 + 3\,\frac{t}{r} + \frac{t^2}{r^2}} \cdot r.$$

$$= \frac{3\pi}{16} \cdot \frac{4}{3}\,r, \text{ when } t \text{ is (as supposed) indefinitely small,}$$

$$= \frac{\pi}{4}\,r, \text{ which is Result (42), Art. 52 of Text.}$$

Fig. 6.



* Rankine's Applied Mechanics, Art. 88, Case XVI.

APPENDIX E.

Sir B. BAKER'S TABLES OF THE WEIGHTS OF IRON AND STEEL BRIDGES,

SHOWING GROSS WEIGHT IN CWTS. PER FOOT RUN OF MAIN SPAN.*

TABLE I.—Short Span Bridges of Wrought-Iron.

Span in Feet.	Bowstring Girders, or Lattice of special construction.	*Continuous.* (a). Two main girders, with cross girders secured to lower flanges of main girders.	*Continuous.* (b). Two main lattice girders of ordinary construction, with cross girders secured as in (a).	*Continuous.* (c) Two main lattice girders of bowstring form with cross girders secured as in (a).	*Lattice.* (a). Two main girders, with cross girders secured to lower flanges of main girders.	*Lattice.* (b). Three main girders, with cross girders secured as in (a).	*Plate Girders.* (a). Two main girders, with cross girders secured to lower flanges of main girders.	*Plate Girders.* (b) Three main girders, with cross girders secured as in (a).	*Plate Girders.* (c). Two main girders, with cross girders secured to upper flanges of main girders, platform brackets, handrail, &c.	*Plate Girders.* (d). Three main girders, with cross girders secured as in (c), platform brackets, &c.	*Plate Girders.* (e). Four main girders under rails, with bracing, platform brackets, &c., but no cross girders.	Remarks.
10						7·2		7·3		5·5	4·7	These weights will be found to correspond to average values of s_t and s_c only. If special values be employed, the weights must be correspondingly altered.
20	9·6	9·9	9·7	9·5	9·8	8·3	10·1	8·6	8·2	6·8	6·2	
30	10·4	10·9	10·7	10·2	10·8	9·4	11·1	10·0	9·2	8·2	7·8	
40	11·2	12·0	11·8	10·9	11·8	10·6	12·3	11·4	10·4	9·9	9·5	
50	12·0	13·0	12·7	11·6	12·8	11·8	13·5	12·9	11·6	11·1	11·1	
60	12·8	14·1	13·7	12·4	13·8	12·8	14·7	14·2	12·8	12·4	12·8	
70	13·7	15·1	14·5	13·1	14·8	14·0	15·9	15·6	14·0	13·8	14·3	
80	14·5	16·2	15·4	13·8	15·8	15·1	17·2	17·1	15·1	15·2	16·0	
90	15·3	17·2	16·3	14·5	16·8	16·2	18·3	18·5	16·3	16·7	17·8	
100	16·1	18·2	17·1	15·2	17·8	17·4	19·5	20·1	17·5	18·3	19·8	
120	17·8	20·2	18·8	16·5	19·8		21·9		19·8			
140	19·4	22·2	20·8	17·9	21·8		24·3		22·1			
160	21·1	24·1	22·2	19·2	23·8		26·7		24·4			
180	22·7	25·8	23·6	20·4	25·9		29·1		26·8			
200	24·4	27·4	25·0	21·5	27·9		31·5		29·1			
225	26·4	29·8	27·0	22·9	30·7		35·0		32·5			
250	28·8	33·3	29·6	24·6	34·2		39·6		37·1			
275	31·8	37·2	32·7	26·8	38·5		45·2		42·7			

* Baker's *Long Span Railway Bridges*, 1873.

TABLE II.—LONG SPAN BRIDGES OF WROUGHT-IRON AND STEEL.

Span in Feet.	Arched Ribs with braced spandrils.		Box-plate Girders (including tubular bridges)		Bowstring Girders (including Saltash type)		Cantilever Lattice of uniform depth.		Cantilever Lattice of varying economic depth		Continuous Lattice of varying depth.		Lattice Girders with parallel flanges.		Straight Link with boom, Bollman type.		Straight Link Suspension		Suspension with Lattice stiffening girder.		Suspended Girders.		Span in Feet	Remarks
	Iron.	Steel	Iron.	Steel.	Iron	Steel.	Iron	Steel.	Iron.	Steel	Iron	Steel	Iron	Steel.	Iron	Steel.	Iron	Steel	Iron.	Steel	Iron	Steel		
300	36	21	55		37	24	29	22	25	19	25	19	38	25	32	21	21	14	26	17	26	17	300	These weights will be found to correspond to average values of s_t and s_c. If special values of s_t and s_c be employed, the weights must be correspondingly altered.
400	44	28	93		56	33	43	31	35	24	35	24	60	38	52	29	30	18	36	24	36	22	400	
500	60	36	156		80	45	59	42	47	33	47	33	96	52	102	40	40	23	50	31	49	28	500	
600	80	46	331		120	59	80	54	62	41	60	41	179	83	164	55	53	30	65	39	64	36	600	
700	100	57	1,458		190	78	109	70	82	48	76	48	496	126	593	78	78	38	87	49	85	44	700	
800	135	71			322	102	157	91	107	64	91	63		225		123	114	49	116	60	113	54	800	
900	189	84			920	140	225	116	142	76	110	75		607		214	200	60	158	74	152	66	900	
1,000	268	102				184	334	152	191	98	128	84		∞		1,304	436	78	211	91	204	81	1,000	
1,100	408	125				238	578	198	265	124	144	96				∞		110	322	113	308	102	1,100	
1,200	701	163				480	1,469	280	388	158	169	109						160	515	148	500	124	1,200	
1,300	1,850	203				1,047		397	705	195	199	123						368	1,143	175	1,120	152	1,300	
1,400		255				∞		650	1,380	242	230	137						597		228		186	1,400	
1,500		333						1,288		315	267	152						2,000		289		234	1,500	
1,600		500						7,000		481	312	168						∞		402		307	1,600	
1,700		735						∞		720	366	189								600		500	1,700	
1,800		1,240								1,326	449	210								1,055		961	1,800	
1,900		3,836								7,150	546	237								2,910		2,320	1,900	
2,000		∞								∞	715	264								∞		∞	2,000	
2,100											981	297											2,100	
2,200											1,375	332											2,200	
2,300											2,000	385											2,300	
2,400												424											2,400	
2,500												506											2,500	
2,600												585											2,600	
2,700												716											2,700	
2,800												861											2,800	
2,900												1,141											2,900	
3,000												1,625											3,000	
3,100												2,821											3,100	
3,200												7,000											3,200	
4,000												∞											4,000	

Baker's *Long Span Railway Bridges*, 1873.

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